Explicit Solutions for a Class of Nonlinear PDE that Arise in Allocation Problems

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Abstract

To exploit large deviation approximations for allocation and occupancy problems one must solve a deterministic optimal control problem (or equivalently, a calculus of variations problem). As this paper demonstrates, and in sharp contrast to the great majority of large deviation problems for processes with state dependence, for allocation problems one can construct more-or-less explicit solutions. Two classes of allocation problems are studied. The first class considers objects of a single type with a parameterized family of placement probabilities. The second class considers only equally likely placement probabilities, but allows for more than one type of object. In both cases, we identify the Hamilton-Jacobi-Bellman equation whose solution characterizes the minimal cost, explicitly construct solutions, and identify the minimizing trajectories. The explicit construction is possible because of the very tractable properties of the relative entropy function with respect to optimization.

1 Introduction

Allocation and occupancy problems are concerned with the random placement of objects into containers. The objects (usually referred to as balls or tokens) can be of a single type or many, in which case they are often distinguished by "color." The containers are variously called urns or cells, and have many interpretations, such as physical partitions (photo-electric receptors in a grid) and temporal partitions (the days of the year).

There are also many rules for how a given ball may be assigned to a given cell. The simplest such rule, in the context of a single color, uses what are called Maxwell-Boltzmann (MB) statistics. Here, each cell is equally likely to receive each

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ball. Other rules consider the balls as being placed sequentially, and the likelihood that a given ball is placed in a given cell depends on the current contents of that cell (relative to the contents of all other cells). Examples in this category are Bose-Einstein statistics (BE), for which a cell that already contains balls is more likely to receive the next ball, and Fermi-Dirac statistics (FD), where the reverse holds. The precise definitions of BE and FD will be given below.

A key random variable associated with an allocation is the *empirical measure*. After all (or some) of the balls have been placed, one can form the (random) probability measure $(\eta_0, \eta_1, \ldots)$ on $\{0, 1, \ldots\}$, with η_0 equal to the fraction of cells that are empty, η_1 the fraction that contain 1, etc. For example, one could be particularly concerned that at least 90% of the cells are nonempty after the random allocation. In this case one is interested in the distribution of the first component of the empirical measure, and in particular $P\{\eta_0 \leq 0.1\}$.

While methods from combinatorial probability provide exact formulas for certain classes of allocation problems, they do not apply universally, nor are they always of great practical utility–see the discussion in [2] on this point. Hence one turns to approximations. The simplest approximation is a law of large numbers (LLN) limit, under which the number of cells and number of balls placed into the cells both tend to ∞ with some fixed ratio. If η is indexed by the number of cells n, then the LLN limit identifies the (deterministic) probability distribution that η^n tends to as $n \to \infty$. This identifies the "typical" behavior of the allocation scheme for large n. The limit can often be identified as the solution to a system of ordinary differential equations (ODEs) at time t (and for an appropriate initial condition), where t is limiting ratio of the number of balls to the number of cells, i.e., the mean number of balls per cell.

If in contrast one is concerned with probabilities of atypical behavior, then one considers large deviation asymptotics. For example, if it is usual that 50% of the cells are empty when n is large, then under some technical assumptions large deviation asymptotics assert that $-\frac{1}{n} \log P\{\eta_0 \leq 0.1\}$ tends to some constant $c > 0$, thus identifying the exponential rate of decay of the probability. The parameter c is usually identified as the solution of a calculus of variations problem, and using the well known relation between problems in calculus of variations and Hamilton-Jacobi equations, c can also be characterized as the value (at a particular point) of the solution to a nonlinear partial differential equation (PDE).

The explicit identification of c is in general a daunting task. Whilst there are a small number of cases for which analytic expressions are available, in most cases one must attempt numerical approximation, and so one is limited to only low dimensional problems (i.e., in our setting to the first few components of the empirical distribution). Even putting aside the restriction of numerical methods to low dimensions, one would prefer analytic expressions for c since they have many other uses. Beyond simply identifying the rate of decay, analytic expressions for c can be used

- to characterize the most likely way that a rare event will occur,

- to construct efficient Monte Carlo schemes (known as importance sampling

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schemes) for non-asymptotic approximations, and

- in statistical estimation and model inference for occupancy models.

The purpose of the present paper is to show that explicit solutions can be obtained for the PDEs that are associated with a wide variety of allocation problems, and introduce techniques that can be applied to even broader classes of problems. As remarked previously, explicit solutions are not common. Among the classes of nonlinear, first order PDE with explicit solutions (in general dimension) are those associated with the linear quadratic regulator and those linked to the Hopf-Lax formula. Both these examples exploit some significant underlying simplification. In the first example it is the fact that the value function for the control problem is expected to be quadratic in the spatial variable, and in the second example it is the independence of the running cost from the state variable. The optimization problems related to allocation problems are qualitatively quite different from either of these, as can be seen from both the form of the value functions and the structure of the minimizing trajectories. There is significant state dependence, and no a priori obvious form for the value function. In the setting of allocation problems, it seems that the attractive properties of the *relative entropy* function are largely responsible for the existence of explicit solutions. It is these properties which allow for convenient calculation and representation of the various derivatives in terms of Lagrange multipliers, the key ingedient in the proof.

In the next section we analyze the single color model. After introducing the general model and formally reviewing the large deviation context, we discuss a formal and heuristic derivation of the explicit solution. The associated Hamilton-Jacobi-Bellman (HJB) equation is then introduced, and a solution is proposed in the form of a finite dimensional minimization problem that can be easily and efficiently solved using Lagrange multiplier techniques. The value of the minimization problem is shown to be smooth for an appropriate class of terminal costs, its derivatives are characterized via multipliers, and the HJB equation is shown to hold. The section concludes with the identification of the minimizing trajectories. The third and final section repeats these steps for a model with different colors.

2 Allocation Models with Differing Assignment Probabilities

2.1 Probabilistic Background and the Variational Problem

In this section, we formulate a general single color occupancy problem. After describing the model, we outline the relevant large deviation properties on path space and the related variational problems.

In the occupancy problem considered here cells are distinguished according to the number of balls contained therein. The full collection of models will be indexed by a parameter a. This parameter takes values in the set $(0, \infty] \cup \{-1, -2, \ldots\}$, and its interpretation is as follows. Suppose that a ball is about to be thrown, and that any two cells (labeled say A and B) are selected. An cell is said to be of *category i*

if it contains i balls. Suppose that cell A is of category i, while B is of category j. Then the probability that the ball is thrown into cell A, conditioned on the state of all the cells and that the ball is thrown into either cell A or B, is

$$\frac{a+i}{(a+i)+(a+j)}.$$

When $a = \infty$ we interpret this to mean that the two cells are equally likely. Also, when $a < 0$ we use this ratio to define the probabilities only when $0 \leq i \vee j \leq -a$ and $i < -a$ or $j < -a$, so the formula gives a well defined probability. The probability that a ball is placed in an cell of category $-a$ is 0. Thus under this model, cells can only be of category $0, 1, \ldots, -a$, and we only throw balls into categories $0, 1, \ldots, -a - 1$.

When $a \in (0, \infty)$ cells that already contain balls are more likely to receive the next ball. When $a < 0$ the opposite is true. The cases $a = 1$, $a = \infty$, $a \in -\mathbb{N}$ correspond to what were called Bose-Einstein statistics, Maxwell-Boltzmann statistics, and Fermi-Dirac statistics, respectively, in the Introduction.

Suppose that before we throw a ball there are already tn balls in all the cells, and that the occupancy state is $(x_0, x_1, \ldots x_{I+})$. Here $x_i, i = 0, 1, \ldots, I$ denotes the fraction of cells that contain i balls, and x_{I+} denotes the fraction containing more than I balls. Throughout this paper we use this convention so that the state space of the occupancy process is finite dimensional. (Explicit formulas analogous to the ones derived here also hold in the infinite dimensional case, though one must be more careful in defining the PDE.) When the occupancy state is $(x_0, x_1, \ldots x_{I+})$, the "un-normalized" or "relative" probability of throwing into a category i cell with $i \leq I$ is simply $(a + i)x_i$. Let us temporarily abuse notation, and let $x_{I+1}, x_{I+2}, \ldots$ denote the exact fraction in each category i with $i > I$. Since there are tn balls in the cells before we throw, $\sum_{i=0}^{\infty} ix_i = t$. Thus the (normalized and true) probability that the ball is placed in an cell that contains exactly i balls, $i = 0, 1, \ldots I$, is $\frac{a+i}{a+t}x_i$, and the probability that the ball is placed in an cell that has more than I balls is $1 - \sum_{j=0}^{I} \frac{a+j}{a+t}x_j$.

In order to define both the LLN and large deviation approximations, it is convenient to introduce an occupancy *process*. We introduce a time variable t that ranges from 0 to T. At a time t that is of the form l/n, with $0 \leq l \leq \lfloor nT \rfloor$ an integer, l balls have been thrown. Let $X^n(t) = \left\{ X_0^n(t), X_1^n(t), \ldots X_I^n(t), X_{I+}^n(t) \right\}$ be the *occupancy state* at that time. As noted previously, $X_i^n(t)$ denotes the fraction of cells that contain i balls at time t, $i = 0, 1, \ldots I$, and $X_{I+}^n(t)$ the fraction of cells that contain more than I balls. The definition of X^n is extended to all $t \in [0, T]$ not of the form l/n by piecewise linear interpolation. Note that $X^n(t)$ is indeed a probability vector in $\mathbb{R}^{I+2}$. If

$$\mathcal{S}_I \doteq \left\{ x \in \mathbb{R}^{I+2} : x_i \geq 0, 0 \leq i \leq I + 1 \text{ and } \sum_{i=0}^{I+1} x_i = 1 \right\},$$

then for any $t \in [0, T]$, $X^n(t) \in \mathcal{S}_I$. Thus X^n takes values in $\mathcal{U} \doteq C([0, T], \mathcal{S}_I)$. We equip $\mathcal{U}$ with the usual supremum norm and on $\mathcal{S}_I$ we take the usual L_1 norm.

It is often the case that one is interested in the large deviation properties at the terminal time T (i.e., those of $X^n(T)$), and for a general initial condition of the form $X^n(t) = (x_0, \ldots, x_{I+})$. Here there is often a detour–one first identifies the large deviation properties of the *process*, and then solves for the large deviation properties of $X^n(T)$ via the so-called Contraction Mapping Theorem. This theorem represents the sought after exponential rate of decay as the solution to a calculus of variations problem, and therein lies the link to a PDE.

For our purposes an informal description of the process level large deviation properties will suffice. We first define the *rate function* on path space. Given (x, t) and a continuous trajectory φ with $\varphi(t) = x$, the rate $\mathcal{I}(\varphi; x, t)$ identifies the decay rate for the probability that X^n is in a small neighborhood of φ:

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} -\frac{1}{n} \log P \left\{ \sup_{t \le s \le T} |X^n(s) - \varphi(s)| < \delta \, \middle| \, X^n(t) = x^n \right\}$$

$$= \lim_{\delta \downarrow 0} \liminf_{n \to \infty} -\frac{1}{n} \log P \left\{ \sup_{t \le s \le T} |X^n(s) - \varphi(s)| < \delta \, \middle| \, X^n(t) = x^n \right\}$$

$$= \mathcal{I}(\varphi; x, t).$$

Here x^n is any sequence of initial conditions that can occur with positive probability and which satisfy $x^n \to x$ as $n \to \infty$. The proof of such a result and the identification of the rate function are given in [7]. $\mathcal{I}(\varphi; x, t)$ can be represented as the integral, over $[t, T]$, of a non-negative "cost" which measures the likelihood that the increments of X^n follow the increments of φ, with higher cost corresponding to lower likelihood (the LLN trajectory has zero cost). The integral form of $\mathcal{I}(\varphi; x, t)$ is a consequence of the Markov property.

The specific form of $\mathcal{I}(\varphi; x, t)$ is as follows. Define the linear map $M : \mathcal{S}_I \mapsto \mathbb{R}^{I+2}$

$$M_i[\theta] = \begin{cases} -\theta_0 & i = 0 \\ \theta_{i-1} - \theta_i & 1 \le i \le I \\ \theta_I & i = I+1 \end{cases}.$$

Let $\varphi \in \mathcal{U}$ be given with $\varphi(t) = x$. Suppose there is a Borel measurable function $\theta : [t, T] \longmapsto \mathcal{S}_I$ such that for any $s \in [t, T]$

$$\varphi(s) = \varphi(t) + \int_t^s M[\theta](u) du. \tag{2.1}$$

We interpret $\theta_i(s)$ as the rate at which balls are thrown into cells that contain i balls at time s. This rate will be viewed as a perturbation of the LLN limit rate at which balls would be thrown, and the cost for this perturbation will measure the likelihood that sure a perturbation occurs (with large cost corresponding to unlikely perturbations).

The mapping $M[\theta]$ accounts for the fact that when a ball is placed in a category i cell X_i^n decreases by $1/n$ and X_{i+1}^n increases by $1/n$. The rates $\theta(s)$ are unique in the sense that if another $\tilde{\theta} : [t, T] \longmapsto \mathcal{S}_I$ satisfies (2.1) then $\tilde{\theta} = \theta$ a.e. on $[t, T]$. We call φ a *valid* occupancy state process if there exists $\theta : [t, T] \longmapsto \mathcal{S}_I$ satisfying (2.1).

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In this case θ is called the occupancy rate process associated with φ. For $x \in \mathbb{R}^{I+2}$ and $t \in [0, -a1_{\{a<0\}} + \infty 1_{\{a>0\}})$, define the vector $\rho(t,x) \in \mathbb{R}^{I+2}$ by

$$\rho_k(t,x) = \frac{a+k}{a+t} x_k, \qquad \text{for } k = 0, 1, \dots I, \tag{2.2}$$

and

$$\rho_{I+}(t,x) = 1 - \sum_{k=0}^{I} \frac{a+k}{a+t} x_k.$$

For each a, $\rho_k(t,x)$ gives the LLN limiting probability that at time t the next ball will be placed in a category-k cell, given that the statistics of model a are used and that $X^n(t) = x$. A direct calculation shows that if

$$x \in \mathcal{S}_I \quad \text{and} \quad \sum_{k=0}^{I+1} k x_k \leq t, \tag{2.3}$$

then $\rho(t,x)$ is indeed a probability vector in $\mathbb{R}^{I+2}$, i.e., $\rho(t,x) \in \mathcal{S}_I$. It is easy to observe that if φ is valid then $\varphi(s)$ satisfies (2.3) for all $s \in [0,T]$. This shows that $\rho(s, \varphi(s)) \in \mathcal{S}_I$. For future use we define

$$\tau(x,t) \doteq \left(t - \sum_{k=0}^{I} k x_k \right) \Big/ x_{I+} \tag{2.4}$$

if $x_{I+} > 0$ and $\tau(x,t) \doteq I+1$ if $x_{I+} = 0$. Thus $\tau(x,t)$ can be interpreted as the mean number of balls per cell among those of category $I+$. With this notation

$$\rho_{I+}(x,t) = (a + \tau(x,t)) x_{I+1}/(a+t), \tag{2.5}$$

and so $\rho_{I+}(x,t)$ in some sense takes a form very similar to that of $\rho_k(x,t)$ for $k = 0, 1, \dots, I$.

Let $\delta > 0$ be small. Observe that the occupancy state will not change very much over $[t, t+\delta]$ while $n\delta$ balls are placed into cells. Let θ denote the empirical measure on the categories where these balls are placed. Then the new occupancy state is the sum of the old state plus $\delta M[\theta]$. Since the change in state is determined by an empirical distribution for (at least approximately) iid random variables, Sanov's Theorem [1, Theorem 2.2.1] suggests that the cost appearing in the integral representation for $\mathcal{I}(\varphi; x, t)$ should be defined in terms of the famous *relative entropy* function. For two probability measures α and β on a Polish space $\mathcal{A}$, the relative entropy of α with respect to β is defined by

$$R\left(\alpha||\beta\right) \doteq \int_{\mathcal{A}} \left(\log \frac{d\alpha}{d\beta} \right) d\alpha$$

whenever α is absolutely continuous with respect to β (and with the convention that $0 \log 0 = 0$). In all other cases we set $R\left(\alpha||\beta\right) = \infty$. When two probability vectors ρ

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and $\nu \in \mathcal{S}_I$ appear in the relative entropy function, we interpret them as probability measures on the simplex $\{0, 1, \ldots, I, I+1\}$, and thus

$$R\left(\rho||\nu\right) \doteq \sum_{i=0}^{I+1} \rho_i \log \frac{\rho_i}{\nu_i}.$$

Important properties of relative entropy are that it is nonnegative, jointly convex and lower semicontinuous in (α, β), and $R\left(\alpha||\beta\right) = 0$ if and only if $\alpha = \beta$ [1, Lemma 1.4.3].

As observed before, when $\varphi(s)$ is *valid,* $\rho\left(s, \varphi(s)\right) \in \mathcal{S}_I$, which makes $R(\theta(s)||\rho\left(s, \varphi(s)\right))$ well defined. If in addition $\varphi(t) = x$, define

$$\mathcal{I}(\varphi; x, t) = \int_t^T R(\theta(s)||\rho\left(s, \varphi(s)\right))ds. \tag{2.6}$$

If φ is not valid or $\varphi(t) \neq x$ then define $\mathcal{I}(\varphi; x, t) = \infty$.

This defines the rate function for the models introduced at the beginning of this section. Now suppose that one wishes to approximate probabilities involving $X^n(T)$. Since the probability that X^n (as a process) is close to a given trajectory φ decays exponentially, decay rates of quantities such as $P\left\{X^n(T) \in A|\, X^n(t) = x^n\right\}$ can (under appropriate regularity conditions on A) be found as follows. Among all trajectories φ with $\varphi(t) = x$ and $\varphi(T) \in A$, identify the one with the *smallest* decay rate c. Then c is also the exponential decay rate of $P\left\{X^n(T) \in A|\, X^n(t) = x^n\right\}$. Hence the variational problem to be solved is

$$V(x, t) = \inf_{\varphi:\varphi(t)=x \text{ and } \varphi(T) \in A} \mathcal{I}(\varphi; x, t). \tag{2.7}$$

If one is interested in expected values other than probabilities then variational problems of the more general form

$$V(x, t) = \inf_{\varphi:\varphi(t)=x} \left[\mathcal{I}(\varphi; x, t) + F(\varphi(T))\right] \tag{2.8}$$

arise, and one is often particularly interested in the initial condition that corresponds to starting with all cells empty: $t = 0, x_0 = 1$ and $x_k = 0, k > 0$. We will refer to this as the *empty* initial condition.

Not all initial conditions are *feasible*, in the sense that they can be reached with finite cost from the empty initial condition. Feasibility in this context depends on the underlying parameter a.

Definition 2.1 (Feasible Domain). *Define $\mathcal{D}_a$, the feasible domain for the occupancy model with parameter a, as follows:*

- *when $a > 0$,*

$$\mathcal{D}_a \doteq \left\{(x, t) \in \mathcal{S}_I \times [0, T] : x_{I+1} > 0 \text{ and } t > \sum_{i=0}^{I+1} i x_i\right\}$$

$$\bigcup \left\{(x, t) \in \mathcal{S}_I \times [0, T] : x_{I+1} = 0 \text{ and } t = \sum_{i=0}^{I} i x_i\right\};$$

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- *and when $a < 0$ and $I = -a - 1$,*

$$\mathcal{D}_a \doteq \left\{ (x, t) \in \mathcal{S}_I \times [0, T) : t = \sum_{i=0}^{I+1} i x_i \right\}.$$

In the first case the second set in the union reflects the fact that when $x_{I+1} = 0$ the number of balls thrown is exactly $\sum_{i=0}^{I} i x_i$, and similarly for the second case. When $a \in -\mathbb{N}$ it is only possible to throw balls into the categories $0, 1, \ldots, -a - 1$, and the only possible categories are $0, 1, \ldots, -a$. Thus if there are n cells there can at most be $-an$ balls thrown, and therefore $T \leq -a$. When $T = -a$ all the cells have exactly $-a$ balls, which is not an interesting case to study. As a consequence, throughout this paper we assume $T < -a$. Also, because of the restriction on the possible categories we can (without loss) assume that $I = -a - 1$. Hence for $a < 0$ we assume without loss that

$$T < -a, \quad I = -a - 1. \tag{2.9}$$

2.2 LLN Limits and Formal Derivation of the Explicit Solution

When constructing explicit solutions one needs some insight into the form of the solution. In this section we present a formal derivation of an explicit solution to (2.8) for the case $F(x) = 1_y(x) \cdot \infty$. Before doing so we calculate the LLN limits of the occupancy processes, a necessary ingredient in the solution.

Equations for the LLN limits can easily be derived directly, or alternatively by noting that they are the zero cost trajectories in the variational problem (2.8) with $F \equiv 0$. It will suffice to consider initial conditions of the form $x = e_k, k = 0, 1, \ldots, I$, where $(e_k)_j$ is 1 if $j = k$ and zero otherwise. Since the relative entropy vanishes only if $\theta(s) = \rho(s, \varphi(s))$, the LLN limits can be characterized by the system of ODEs

$$\dot{\varphi}(s) = M[\rho(s, \varphi(s))], \quad \varphi(t) = e_k. \tag{2.10}$$

Since the LLN limit is desired for all components of the occupancy process, we use the infinite system rather than the system truncated at $I+$. These are easy to solve because the equation for the jth component depends only on the $j - 1$st component, and so one can solve first for the kth component and then bootstrap. To write the solution in explicit form, we need some notation. For all $a \in \mathbb{R}$, $a \neq 0$ and $i \in \mathbb{N}$, let

$$\binom{a}{i} \doteq \frac{\prod_{j=0}^{i-1}(a - j)}{i!}.$$

Note that if $a \in \mathbb{N}$ and $i > a$ then $\binom{a}{i} = 0$, and that if $a \notin \mathbb{N} \cup \{0\}$ and $i \in \mathbb{N}$, then $\binom{a}{i} \neq 0$. For $i \in \mathbb{N} \cup \{0\}$ and $a > 0, s \geq 0$ or $a \in -\mathbb{N}, 0 \leq s \leq -a$, define

$$\mathcal{Q}_i^a(s) \doteq \left(-\frac{s}{a}\right)^i \binom{-a}{i} \left(1 + \frac{s}{a}\right)^{-a-i}.$$

One can easily check that the solution to (2.10) is $\varphi_i(s) = 0$ if $i < k$, and

$$\varphi_{k+i}(s) = \mathcal{Q}_i^{a+k} \left(\frac{a + k}{a + t}(s - t) \right)$$

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if $k \geq i$. In the limit $a \to \infty$ (MB statistics) one obtains the Poisson distribution

$$\mathcal{Q}_i^{a+k} \left(\frac{a+k}{a+t}(s-t) \right) \to \mathcal{P}_i(s-t) = e^{-(s-t)}(s-t)^i/i!.$$

For the remainder of this section we assume $a \neq \infty$, with the understanding that analogous statements for $a = \infty$ can be obtained by passing to the limit.

We next present a formal and heuristic solution to the variational problem based on probabilistic intuition. Recall that the variational problem is intended to approximate the normalized logarithm of a probability. If one decomposes a probability into products or conditional products, this will correspond to a decomposition of the quantity being minimized as a sum.

We wish to solve (2.7) when $A = \{y\}$. Suppose that x_i is interpreted as the size of the "pool" of cells that start at time t in category i. Through the random placements, this pool will evolve into sub-pools of differing categories. Let π_i^k denote the probability that a cell of category k at time t ends up a cell of category $k+i$ at time T. Then satisfaction of the terminal constraint requires

$$y_i = \sum_{k=0}^{i} x_k \pi_{i-k}^k, \quad 0 \leq i \leq I, \qquad y_{I+1} = 1 - \sum_{k=0}^{I} y_k.$$

We use $y \doteq x \times \pi$ as shorthand for the last display. We require that the π^k be probabilities, and also a constraint that corresponds to the fact that $n(T-t)$ balls will be placed in the prelimit problem:

$$x_k \sum_{j=0}^{\infty} \pi_j^k = x_k, \quad 0 \leq k \leq I+1, \quad \sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j\pi_j^k = T - t, \tag{2.11}$$

Let $\mathcal{F}(x, t; y, T)$ denote the set of $\pi = (\pi^0, \pi^1, \ldots, \pi^I, \pi^{I+1})$ which satisfy the last two displays. A terminal point y is *feasible* (for the given initial time and condition) if $\mathcal{F}(x, t; y, T)$ is not empty.

To guess the form of the solution to the variational problem, we consider the allocation from a different perspective. Owing to the fact that the un-normalized relative probabilities are *affine* in the number of balls currently in each cell, we can first study the random evolution of the number of balls that are in each pool. This can be done without knowing the details of how the balls are placed *within* the pool. Indeed, the structural properties of the placement probabilities imply that this process is also Markovian, and its large deviation properties are easy to identify.

Once we know the total number of balls that will end up in each pool, we then consider the question of how they are distributed among cells within the pool. Here we make an approximation that is formal but reasonable, which is that the rate function for the empirical distribution within the pool can be found as follows. We first generate iid random variables according to the LLN distribution appropriate to the particular pool, and form the empirical distribution for this sample. The rate function for this empirical distribution is a certain relative entropy identified by Sanov's Theorem. However, we must also impose the constraint on the (previously

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determined) number of balls that were placed into this particular pool, which adds a constraint to the rate function. Finally, the overall rate function is found by combining these two rates. The function found by this manner will be proved to be the solution to the calculus of variations problem.

An argument based on Sanov's Theorem shows that the variational problem for the allocation between the pools is

$$\inf \int_t^T R\left(u(s)\,\|\,w(s)\right)ds,$$

where

$$w_k(s) = \frac{(a+k)x_k + \int_t^s u_k(\tau)d\tau}{a+s}$$

is the probability that a ball is placed into pool k at time s. This is, in un-normalized form, equal to $ax_k + [\text{number of balls per cell in pool } k]x_k$, and the normalization is just $a+s$. The initial and terminal conditions are

$$w_k(t) = \frac{(a+k)x_k}{a+t}, \quad w_k(T) = \frac{(a+k)x_k + z_k(T-t)}{a+T},$$

where z_k is the mean number of additional balls per unit time put into pool k. The Euler-Lagrange equations for this problem are easily constructed and solved, and one obtains as the optimal trajectory

$$w_k(s) = \frac{1}{s+a}\left((a+k)x_k + (s-t)z_k\right)$$

(of course satisfaction of the Euler-Lagrange equations is not in general a sufficient condition for optimality, but since our discussion is simply to motivate the form of the solution this point is of no consequence). The cost is

$$\int_t^T R\left([(s+a)w(s)]'\,\|\,w(s)\right)ds,$$

and for the optimal trajectory

$$[(s+a)w_k(s)]' = z_k.$$

The integral can be explicitly evaluated, and equals

$$\sum_{k=0}^{I+} z_k \cdot (T-t) \cdot \log\left(\frac{a+t}{a+k} \cdot \frac{z_k}{x_k}\right)$$

$$-\sum_{k=0}^{I+} [x_k \cdot (a+k) + z_k \cdot (T-t)] \cdot \log\left(\frac{a+k+\frac{z_k}{x_k}(T-t)}{a+k+\frac{a+k}{a+t}(T-t)}\right).$$

This identifies the first part of the overall rate function.

The second part is found by considering placement within each pool. The mean additional number of balls per cell in pool k is $(z_k/x_k)(T-t)$. According to the

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LLN, the number of additional balls in a typical cell from pool k has distribution $\mathcal{Q}^{a+k}\left(\frac{z_k}{x_k}(T-t)\right)$ if $k \leq I$ and $\mathcal{Q}^{a+\tau(x,t)}\left(\frac{z_{I+}}{x_{I+}}(T-t)\right)$ if $k = I+$. Approximating the true empirical measure within a given pool by that of the empirical measure for iid random variables with the corresponding distribution, one formally obtains from Sanov's theorem the rate function $R\left(\pi^k \left\| \mathcal{Q}^{a+k}\left(\frac{z_k}{x_k}(T-t)\right)\right.\right)$, together with the constraint $\sum_{i=0}^{\infty} i\pi_i^k = \frac{z_k}{x_k}(T-t)$ on the number of balls placed in pool k. Combining the different contributions from the various pools with the contribution due to the allocation between the pools and then applying the terminal constraint, one (again formally) obtains the rate function

$$\inf \left\{ \sum_{k=0}^{I} x_k R\left(\pi^k \left\| \mathcal{Q}^{a+k}\left(\frac{z_k}{x_k}(T-t)\right)\right.\right)\right.$$

$$+ x_{I+} R\left(\pi^{I+} \left\| \mathcal{Q}^{a+\tau(x,t)}\left(\frac{z_{I+}}{x_{I+}}(T-t)\right)\right.\right) + \sum_{k=0}^{I+} z_k \cdot (T-t) \cdot \log\left(\frac{a+t}{a+k} \cdot \frac{z_k}{x_k}\right)$$

$$\left. - \sum_{k=0}^{I+} \left[x_k \cdot (a+k) + z_k \cdot (T-t)\right] \cdot \log\left(\frac{a+k+\frac{z_k}{x_k}(T-t)}{a+k+\frac{a+k}{a+t}(T-t)}\right)\right\},$$

where the infimum is over all π and z such that $\sum_{i=0}^{\infty} i\pi_i^k = \frac{z_k}{x_k}(T-t)$ and $x \times \pi = y$. However, a straightforward calculation using the specific form of $\mathcal{Q}^a$ and $\sum_{i=0}^{\infty} i\pi_i^k = \frac{z_k}{x_k}(T-t)$ gives

$$R\left(\pi^k \left\| \mathcal{Q}^{a+k}\left(\frac{a+k}{a+t}(T-t)\right)\right.\right) - R\left(\pi^k \left\| \mathcal{Q}^{a+k}\left(\frac{z_k}{x_k}(T-t)\right)\right.\right)$$

$$= \frac{z_k}{x_k}(T-t) \cdot \log\left(\frac{a+t}{a+k} \cdot \frac{z_k}{x_k}\right) - (a+k) \cdot \log\left(\frac{a+k+\frac{z_k}{x_k}(T-t)}{a+k+\frac{a+k}{a+t}(T-t)}\right)$$

$$- (z_k/x_k)(T-t) \cdot \log\left(\frac{a+k+\frac{z_k}{x_k}(T-t)}{a+k+\frac{a+k}{a+t}(T-t)}\right),$$

with an analogous result for $k = I+$. If follows that the rate function can be written in the simpler form

$$\inf_{\pi:x\times\pi=y}\left\{\sum_{k=0}^{I} x_k R\left(\pi^k \left\| \mathcal{Q}^{a+k}\left(\frac{a+k}{a+t}(T-t)\right)\right.\right)\right.$$

$$\left. + x_{I+} R\left(\pi^{I+} \left\| \mathcal{Q}^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t)\right)\right.\right)\right\},$$

with the infimum over z no longer necessary.

Let

$$\mathcal{J}(x,t;y) \doteq \inf_{\substack{\varphi \in C([t,T],\mathcal{S}_I) \\ \varphi(t)=x, \varphi(T)=y}} \mathcal{I}(x,t;\varphi). \tag{2.12}$$

The formal derivation just given suggests the following result, in which we also simplify further where the special cases of FD and MB statistics allow.

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Theorem 2.2 (Explicit Formula for the Rate Function). *Consider an initial condition $(x,t) \in \mathcal{D}_a$, and a feasible terminal condition y. If $a \in (0,\infty)$, define $\tau(x,t)$ by (2.4). Then the quantity $\mathcal{J}(x,t;y)$ defined in (2.12) has the representation*

$$
\mathcal{J}(x,t;y) = \min_{\pi \in \mathcal{F}(x,t;y,T)} \left\{ \sum_{k=0}^{I} x_k R\left(\pi^k \,\middle\|\, \mathcal{Q}^{a+k}\left(\frac{a+k}{a+t}(T-t) \right) \right) \right.
$$

$$
\left. + x_{I+1} R\left(\pi^{I+1} \,\middle\|\, \mathcal{Q}^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t) \right) \right) \right\}.
$$

If $a \in -\mathbb{N}$ with $I = -a - 1$ then $\tau(x,t) = I + 1$, and

$$
\mathcal{J}(x,t;y) = \min_{\pi \in \mathcal{F}(x,t;y,T)} \left\{ \sum_{k=0}^{I+1} x_k R\left(\pi^k \,\middle\|\, \mathcal{Q}^{a+k}\left(\frac{a+k}{a+t}(T-t) \right) \right) \right\}.
$$

In the final case of $a = \infty$, we have

$$
\mathcal{J}(x,t;y) = \min_{\pi \in \mathcal{F}(x,t;y,T)} \left\{ \sum_{k=0}^{I+1} x_k R\left(\pi^k \,\middle\|\, \mathcal{P}(T-t) \right) \right\}.
$$

Remark 2.3. Although the minimization problems in Theorem 2.2 appear to be infinite dimensional, they can in fact be reduced to finite dimensional problems. This is because if π^k is the minimizer, then π_j^k takes a prescribed form for $j > I$. In fact, all π_j^k can be represented in terms of no more than $I + 3$ Lagrange multipliers as in (2.20) below.

2.2.1 The Hamilton-Jacobi-Bellman equation

Given Theorem 2.2 one can solve the problem with a general terminal condition F. Conversely, if the problem with terminal cost can be solved for a sufficiently broad class of F, one can derive Theorem 2.2. This is how we will prove the theorem, and moreover the proof will be based on the fact that finite dimensional representations analogous to those in Theorem 2.2 but with these terminal costs are *classical sense* solutions to the associated PDE. The proof also has a number of side benefits, such as convenient representations for the various derivatives of the solution in terms of Lagrange multipliers.

The calculus of variations problem (2.8) has a natural control interpretation, where $\theta(s)$, $t \le s \le T$ is the control, $\dot{\varphi}(s) = M[\theta](s)$ are the dynamics, $R(\theta(s)\|\rho(s,\varphi(s)))$ is the running cost and $F(x)$ is the terminal cost. It is expected that if we define

$$
V(x,t) \doteq \inf_{\varphi \in C([t,T],\mathcal{S}_I),\varphi(t)=x} \left\{ \int_t^T R(\theta(s)\|\rho(s,\varphi(s)))\,ds + F(\varphi(T)) \right\}, \quad (2.13)
$$

then $V(x,t)$ is a weak-sense solution to the *Hamilton-Jacobi-Bellman (HJB)* equation

$$
W_t + H(W_x, x, t) = 0,
$$

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and terminal condition
$$W(x, T) = F(x).$$
Here the Hamiltonian $H(p, x, t)$ is defined by

$$H(p, x, t) \doteq \inf_{\theta \in \mathcal{S}_I} \left[\langle p, M[\theta] \rangle + R\left(\theta \, \| \rho\left(t, x \right) \right) \right]$$

and W_t and W_x denote the partial derivative with respect to t and gradient in x, respectively. Note that by the representation formula [1, Proposition 1.4.2], the infimum in the definition of $H(p, x, t)$ can be evaluated, yielding

$$\begin{cases} W_t = \log \left(\sum_{k=0}^I x_k \left(\frac{a+k}{a+t} \right) \exp\left(W_{x_k} - W_{x_{k+1}} \right) + x_{I+1} \left(\frac{a+\tau(x,t)}{a+t} \right) \right) \\ W(x, T) = F(x) \end{cases} . \quad (2.14)$$

Note the use of the convenient expression (2.5) for $\rho_{I+}(x, t)$.

For a general smooth F (2.14) need not have a smooth (C^1) solution. However, for affine terminal costs $F(x) = \langle \ell, x \rangle + b$ there is a C^1 solution (it is in fact the unique solution), and as remarked above, these solutions can be used to carry out a fairly complete analysis of the problem with more general terminal conditions. Indeed, for a general (proper) convex terminal cost $F(x)$, the Legendre transform gives a representation of the form

$$F(x) = \sup_{\beta \in \mathbb{R}^{I+2}} \left[\langle \beta, x \rangle - h(\beta) \right]$$

for some proper convex function h. Let $V^F(x, t)$ denote the solution (explicit or otherwise) to the calculus of variations problem (2.13) with terminal cost $F(\cdot)$. Then one can show
$$V^F(x, t) = \sup_{\beta \in \mathbb{R}^{I+2}} V^{\{\langle \beta, \cdot \rangle - h(\beta)\}}(x, t),$$

and an analogous formula for $U^F(x, t) \doteq \inf[\mathcal{J}(x, t; y) + F(y)]$. Given Proposition 2.4 below, $V^F = U^F$ then follows. Since $\infty \cdot 1_{\{y\}^c}$ is a proper convex function, the formula can be extended even further to very general F.

Observe that W is a solution of just the PDE alone (i.e., without the terminal condition) if and only if $W + c$ is a solution for any real number c. Since x is a probability vector, it suffices to prove the representation under the conditions $\ell_{I+1} = 0$ and $b = 0$.

2.3 Explicit solution for affine terminal costs

Proposition 2.4. *Consider* $(x, t) \in \mathcal{D}_a$ *and* $F(y) = \langle \ell, y \rangle$, *where* $\ell \in \mathbb{R}^{I+2}$ *and* $\ell_{I+1} = 0$. *Define*

$$V(x, t) \doteq \inf_{\varphi \in C([t, T], \mathcal{S}_I), \varphi(t) = x} \left\{ \int_t^T R\left(\theta(s) \| \rho\left(s, \varphi(s) \right) \right) ds + F(\varphi(T)) \right\},$$

and

$$U(x,t) \;\doteq\; \min_{\pi \in \mathcal{F}(x,t;T)} \left\{ \sum_{k=0}^{I} x_k R \left(\pi^k \,\Big\|\, \mathcal{Q}^{a+k}\left(\frac{a+k}{a+t}(T-t) \right) \right) \right. \tag{2.15}$$

$$\left. + x_{I+1} R \left(\pi^{I+1} \,\Big\|\, \mathcal{Q}^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t) \right) \right) + F(x \times \pi) \right\}$$

where $\pi \in \mathcal{F}(x,t;T)$ means that π satisfies the constraints in (2.11). Then $V(x,t) = U(x,t)$.

The proof of this result is given in the next subsection. We close this subsection with remarks on the LLN limit distributions.

We will use the fact that if $a \in \mathbb{R}$ and $|z| < 1$ then the binomial expansion

$$(1+z)^{-a} = \sum_{i=0}^{\infty} \binom{-a}{i} z^i, \qquad \binom{a}{i} \doteq \frac{\prod_{j=0}^{i-1}(a-j)}{i!}$$

is valid, and if $-a \in \mathbb{N}$ then the sum contains only a finite number of nonzero terms and is valid for all $z \in \mathbb{R}$. Recall that for $i \in \mathbb{N} \cup \{0\}$ and $a > 0, s \geq 0$ or $a \in -\mathbb{N}, 0 \leq s \leq -a$, then

$$\mathcal{Q}_i^a(s) \doteq \left(-\frac{s}{a} \right)^i \binom{-a}{i} \left(1 + \frac{s}{a} \right)^{-a-i}.$$

If $a > 0, s \geq 0$ and $|s\theta/(a+s)| < 1$, then the binomial expansion gives

$$\begin{aligned}
\sum_{i=0}^{\infty} \mathcal{Q}_i^a(s)\theta^i &= \sum_{i=0}^{\infty} \left(-\frac{s}{a} \right)^i \binom{-a}{i} \left(1 + \frac{s}{a} \right)^{-a-i} \theta^i \\
&= \left(1 + \frac{s}{a} \right)^{-a} \sum_{i=0}^{\infty} \left(-\frac{s}{s+a}\theta \right)^i \binom{-a}{i} \\
&= \left(1 + \frac{s}{a} \right)^{-a} \left(1 - \frac{s\theta}{s+a} \right)^{-a} \\
&= \left(1 + \frac{s}{a} - \frac{s\theta}{a} \right)^{-a} \\
&= \left(1 + \frac{s}{a}(1-\theta) \right)^{-a}.
\end{aligned}$$

We thus have the following expressions, where the second one may be justified by a very similar calculation (when $|s\theta/(a+s)| < 1$):

$$\sum_{i=0}^{\infty} \mathcal{Q}_i^a(s)\theta^i = \left(1 + \frac{s}{a}(1-\theta) \right)^{-a}, \quad \sum_{i=0}^{\infty} i\mathcal{Q}_i^a(s)\theta^i = s\theta \left(1 + \frac{s}{a}(1-\theta) \right)^{-a-1}. \tag{2.16}$$

Note also that when $-a \in \mathbb{N}$ and $0 \leq s \leq -a$ the number of nonzero summands is finite and the formulas again hold. If $-a \in \mathbb{N}$ and $0 \leq s \leq -a$ or $a > 0$ and $s \geq 0$ then $\mathcal{Q}_i^a(s) \geq 0$ for $i \in \mathbb{N} \cup \{0\}$. Letting $\theta \uparrow 1$ in the first expression shows that

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under these conditions $\mathcal{Q}_i^a(s)$ defines a probability measure on $\mathbb{N} \cup \{0\}$. When $a = 0$ we use the limiting values

$$\mathcal{Q}_0^0(s) = 1, \ \mathcal{Q}_i^0(s) = 0$$

for all $i \in \mathbb{N}$ and $s \geq 0$. For later use note that similar calculations show that if $-a \in \mathbb{N}$ and $0 \leq s \leq -a$ or $a > 0$, $s \geq 0$, and $s/(a+s) < 1$, then

$$\sum_{i=0}^{\infty} i^2 \mathcal{Q}_i^a(s) - \left[\sum_{i=0}^{\infty} i \mathcal{Q}_i^a(s)\right]^2 = s + \frac{s^2}{a}. \tag{2.17}$$

2.3.1 Analysis of the finite dimensional minimization problem

We now focus on proving Proposition 2.4. We will do so by proving that $U(x,t)$ is a classical sense solution to the HJB equation (2.14). A modification of the standard verification argument [4] can then be used to show that $V(x,t) = U(x,t)$. The classical verification argument consists of two parts. One first considers any valid occupancy process and control (φ, θ) for the initial condition (x,t). If U is a smooth solution to the PDE (2.14) in neighborhood of $\{(\varphi(s), s) : t \leq s \leq T\}$ and if $U(\varphi(T), T) = \langle \varphi(T), \ell \rangle$, then the chain rule implies that the cost along this trajectory is at least $U(x,t)$. The reverse inequality is proved by defining an optimal feedback control through the HJB equation, using this control to construct a trajectory, and then verifying (once again via the chain rule) that the cost for this control is $U(x,t)$. The characterization of $V(x,t)$ as an infimum over all valid occupancy processes and controls that start at (x,t) then gives $V(x,t) = U(x,t)$. However, we have to clarify here what is meant by a "classical sense" solution to (2.14). The difficulty is that $U(x,t)$ is only well defined on the set $\mathcal{D}_a$, which does not have interior.

Given any point $(x,t) \in \mathcal{D}_a$, we will prove that one can extend $U(x,t)$ smoothly to a neighborhood of (x,t) in $\mathbb{R}^{I+2} \times \mathbb{R}$. To be more precise, for any such (x,t) we will show there exists a neighborhood $\mathcal{U} \subset \mathbb{R}^{I+2} \times \mathbb{R}$ of (x,t) and a function $\bar{U} \in C^{\infty}(\mathcal{U}, \mathbb{R})$, such that $\bar{U}(y,s) = U(y,s)$ for $(y,s) \in \mathcal{U} \cap \mathcal{D}_a$, and that $\bar{U}$ satisfies (2.14) in $\mathcal{U} \cap \mathcal{D}_a$. One can then use $\bar{U}$ in place of U in the verification argument, since any feasible trajectory will never leave $\mathcal{D}_a$.

To analyze $U(x,t)$ we formulate an appropriate Lagrangian. Let

$$
\begin{aligned}
f(x,t;\pi) \ &\doteq \ \sum_{k=0}^{I} x_k R\left(\pi^k \left\| \mathcal{Q}^{a+k}\left(\frac{a+k}{a+t}(T-t)\right)\right.\right) \tag{2.18} \\
&\quad + x_{I+1} R\left(\pi^{I+1} \left\| \mathcal{Q}^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t)\right)\right.\right) + \langle \ell, x \times \pi \rangle
\end{aligned}
$$

and for a set of Lagrange multipliers $\Lambda \doteq (\lambda, \mu) = (\lambda_0, \lambda_1, \ldots, \lambda_I, \lambda_{I+1}, \mu)$, let

$$
\begin{aligned}
&L(x,t;\Lambda;\pi) \tag{2.19} \\
&\doteq \ f(x,t;\pi) + \sum_{k=0}^{I+1} \lambda_k x_k \left(1 - \sum_{j=0}^{\infty} \pi_j^k\right) + \mu\left(T - t - \sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j\pi_j^k\right).
\end{aligned}
$$

It follows from the definition of $U(x,t)$ that $U(x,t) = \inf_{\pi} \sup_{\Lambda} L(x,t;\Lambda;\pi)$.

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Note that by the joint convexity of relative entropy, $L(x, t; \Lambda; \pi)$ is convex in π. Thus (2.15) is a standard convex programming problem with linear constraints, except that the minimization is over a variable π which is infinite dimensional. Hence the standard Lagrange multiplier method does not apply directly. If we temporarily ignore this issue, then to guess the form of the minimizer one would of course set $D_\pi L(x, t; \Lambda; \pi) = 0$ to get $\pi = \pi(x, t; \Lambda)$, where D_π stands for the gradient in π and

$$
\begin{aligned}
\pi_j^k(x, t; \Lambda) &= \mathcal{Q}_j^{a+k}\left(\frac{a+k}{a+t}(T-t)\right) e^{\lambda_k - 1 + j\mu - \ell_{k+j}} \\
&\qquad k = 0, 1, \ldots, I \text{ and } j \geq 0, \qquad\qquad (2.20) \\
\pi_j^{I+1}(x, t; \Lambda) &= \mathcal{Q}_j^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t)\right) e^{\lambda_{I+1} - 1 + j\mu}.
\end{aligned}
$$

Here, for notational simplicity, we extend ℓ in Proposition 2.4 by letting $\ell_i = 0$ when $i > I$. Note in particular that $\{\pi^k\}$ will depend on x only when $k = I + 1$. Observe also that setting $D_\Lambda L(x, t; \Lambda; \pi) = 0$ gives the constraints (2.11). For any $(x, t) \in \mathbb{R}^{I+2} \times \mathbb{R}$ and $\Lambda \in \mathbb{R}^{I+3}$, let $\pi(x, t; \Lambda)$ be determined by (2.20) and define $G : \mathbb{R}^{I+2} \times \mathbb{R} \times \mathbb{R}^{I+3} \mapsto \mathbb{R}^{I+3}$ by

$$
\begin{aligned}
G_k(x, t; \Lambda) &= \left(1 - \sum_{j=0}^\infty \pi_j^k(x, t; \Lambda)\right), \qquad k = 0, 1, \ldots, I + 1 \\
G_{I+2}(x, t; \Lambda) &= \left(T - t - \sum_{k=0}^{I+1} x_k \sum_{j=0}^\infty j \pi_j^k(x, t; \Lambda)\right).
\end{aligned}
$$

In the next theorem we show that the $\pi(x, t; \lambda)$ defined in (2.20) indeed give the minimizer of (2.15).

Theorem 2.5. *For any $(x, t) \in \mathcal{D}_a$ define $U(x, t)$ by (2.15). Then there exists $\Lambda \in \mathbb{R}^{I+3}$ so that $G(x, t; \Lambda) = 0$, and $\pi(x, t; \Lambda)$ is a minimizer of (2.15). Thus $U(x, t) = L\left(x, t; \Lambda; \pi(x, t; \Lambda)\right)$. In addition, the Λ that satisfies $G(x, t; \Lambda) = 0$ is unique. Hence if $G(x, t; \Lambda) = 0$ for some $\Lambda \in \mathbb{R}^{I+3}$, then $\pi(x, t; \Lambda)$ is a minimizer of (2.15).*

The proof is divided into three lemmas. For a point $(x, t) \in \mathcal{D}_a$, quantities of the following sort will appear frequently in the proofs of the lemmas:

$$
\begin{aligned}
\bar{\pi}_j^k &\doteq \mathcal{Q}_j^{a+k}\left(\frac{a+k}{a+t}(T-t)\right) \qquad\qquad\qquad (2.21) \\
\bar{\pi}_j^{I+1} &\doteq \mathcal{Q}_j^{a+\tau(x,t)}\left(\frac{a+\tau(x,t)}{a+t}(T-t)\right) \\
&\qquad \text{for } k = 0, 1, \ldots, I; j = 0, 1, \ldots.
\end{aligned}
$$

In particular, it will often be the case that (2.16) must be invoked, with a there replaced by $a+k$ [or $a+\tau(x, t)$] and s there replaced by the corresponding argument in the expression above. We note that the conditions required for (2.16) will always

hold so long as $t \in [0, T]$. This is straightforward to check in the case of $a > 0$. For the case $-a \in \mathbb{N}$, it uses that $-a = I + 1$, $T \leq -a$, and that always $\tau(x, t) = I + 1$. Thus for example for any $k \in \{0, \ldots, -a\}$ and $t \in [0, T]$, $(a + k)(T - t)/(a + t) \geq 0$ and $(T - t)/(-a - t) \leq 1$ shows that $(a + k)(T - t)/(a + t) \leq -a - k$, as required for (2.16).

Lemma 2.6 (General properties). *For any $(x, t) \in \mathcal{D}_a$ define $U(x, t)$ by (2.15). Then $\mathcal{F}(x, t; T)$ is nonempty, minimizing measures π^* exist, and if k is such that $x_k > 0$ and $j \in \{0, 1, \ldots\}$, then*

$$\bar{\pi}_j^k > 0 \quad implies \quad \pi_j^{*k} > 0. \tag{2.22}$$

Proof. According to (2.16) the quantities in (2.21) are probabilities that satisfy (2.11). This shows that $\mathcal{F}(x, t; T)$ is nonempty. Next note that with this notation, we can rewrite (2.18) as

$$f(x, t; \pi) = R(x \otimes \pi \,\|\, x \otimes \bar{\pi}) + \langle \ell, x \times \pi \rangle, \tag{2.23}$$

where $(x \otimes \pi)_{i,j} = x_i \pi_j^i$. Since the relative entropy has compact level sets in the first argument [1, Lemma 1.4.3(c)], the existence of a minimizer of (2.15) follows. In addition, because of the strict convexity in that argument we know that the minimizer is unique up to those $\{\pi_\cdot^k\}$ with $x_k > 0$.

For a general initial condition (x, t) let $\mathcal{K} \doteq \{k : x_k > 0\}$. Then the choice of $\{\pi_\cdot^k : k \notin \mathcal{K}\}$ will not affect either the constraint (2.11) or the objective function (2.23). Hence we can consider the equivalent minimization problem over $\mathcal{M}_{(x,t)} = \{\pi_j^k : k \in \mathcal{K}, j = 0, 1, \ldots\}$. As discussed in the previous paragraph, a minimizer in $\mathcal{M}_{(x,t)}$ exists and is unique. Let this minimizer be denoted π^*.

Lastly we must show (2.22). Let $\pi^\epsilon \doteq (1 - \epsilon)\pi^* + \epsilon\bar{\pi}$, where $\bar{\pi}$ is defined in (2.21) and let $f(\epsilon) = f(x, t; \pi^\epsilon)$. By computing the derivative of $f(\epsilon)$ explicitly, it is readily observed that if (2.22) does not hold then $f'(\epsilon) \to -\infty$ as $\epsilon \to 0$. Thus (2.22) must be true since otherwise π^* is not the minimizer. $\qquad\square$

Lemma 2.7 (Characterization of the minimizer). *For any $(x, t) \in \mathcal{D}_a$ define $U(x, t)$ by (2.15). Then there exists $\Lambda \in \mathbb{R}^{I+3}$ so that $G(x, t; \Lambda) = 0$, and $\pi(x, t; \Lambda)$ is a minimizer of (2.15).*

Proof. We want to argue that the minimizers must take the form of (2.20). However, there is a difficulty since $\mathcal{M}_{(x,t)}$ can be infinite dimensional. To deal with this we use a truncation argument adapted from one in [2]. For any $N \in \mathbb{N}$ let

$$T^{(N)} \doteq \sum_{k \in \mathcal{K}} x_k \sum_{j=0}^{N} j\pi_j^{*k}, \quad \alpha_k^{(N)} \doteq \sum_{j=0}^{N} \pi_j^{*k}, \quad k \in \mathcal{K}$$

and also let

$$f^{(N)}(x, t; \pi) \doteq f(x, t; \hat{\pi}),$$

where

$$\hat{\pi}_j^k = \pi_j^k \quad \text{for } k \in \mathcal{K}, j \leq N$$
$$\hat{\pi}_j^k = \pi_j^{*k} \quad \text{for } k \in \mathcal{K}, j > N.$$

Since π^* is the minimizer of (2.15) automatically

$$U(x,t) = \min_\pi f^{(N)}(x,t;\pi), \tag{2.24}$$

where the minimum is subject to the constraints

$$\sum_{k \in \mathcal{K}} x_k \sum_{j=0}^N j\pi_j^k = T^{(N)}, \quad \sum_{j=0}^N \pi_j^k = \alpha_k^{(N)}, \quad k \in \mathcal{K}.$$

We can now apply the standard Lagrange multiplier method to (2.24). The first step is to formulate the Lagrangian for this finite dimensional problem:

$$L^N(x,t;\Lambda;\pi)$$
$$\doteq f^N(x,t;\pi) + \sum_{k \in \mathcal{K}} \lambda_k^{(N)} x_k \left(\alpha_k^{(N)} - \sum_{j=0}^N \pi_j^k \right) + \mu^{(N)} \left(T^{(N)} - \sum_{k=0}^{I+1} x_k \sum_{j=0}^N j\pi_j^k \right).$$

We have that $\left\{ \pi_j^{*k} : k \in \mathcal{K}, j \leq N \right\}$ satisfies the constraints in (2.24), and by (2.22) we know that $\pi_j^{*k} > 0$ if $\bar{\pi}_j^k > 0$. Hence by [6, Corollary 28.2.2] and [6, Theorem 28.3] applied to (2.24), there must exist a set of Lagrange multipliers $\lambda_k^{(N)}, \mu^{(N)}$ so that the minimizer of (2.24) π_j^{*k} has the form

$$\pi_j^{*k} = \bar{\pi}_j^k e^{\lambda_k^{(N)} - 1 + j\mu^{(N)} - \ell_{k+j}} \tag{2.25}$$

for $k \in \mathcal{K}$ and $0 \leq j \leq N$. If $k + j > I$, then since $\ell_{k+j} = 0$

$$\frac{\pi_{j+1}^{*k}}{\pi_j^{*k}} = C \cdot e^{\mu^{(N)}},$$

where C does not depend on N. Thus $\mu^{(N)}$ is independent of N, and hence $\lambda^{(N)}$ is also independent of N. Since the choice of N is arbitrary, we then know that for all $k \in \mathcal{K}$ and $j = 0, 1, ..., \pi_j^{*k}$ indeed has the form in (2.20) for a suitable choice of λ_k and μ. For $k \notin \mathcal{K}$, we can simply define π_j^{*k} as in (2.25) and then solve for λ_k from the normalization constraint $\sum_{j=0}^\infty \pi_j^{*k} = 1$. When defined in this way, $\Lambda^* = (\lambda_0, \ldots, \lambda_{I+1}, \mu)$ automatically satisfies $G(x,t;\Lambda^*) = 0$.

For $k \in \mathcal{K}$ the corresponding λ_k are a Kuhn-Tucker vector as in [6, Corollary 28.2.2], and hence each $\lambda_k < \infty$. However, for $k \notin \mathcal{K}$ the finiteness of λ_k is not automatic.

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To show the finiteness, we first insert the explicit form of $\pi_j^{I+1}(x,t;\Lambda)$ from (2.20) into $G_{I+1}(x,t,\Lambda) = 0$ to obtain

$$\sum_{j=0}^{\infty} \mathcal{Q}_j^{a+\tau(x,t)} \left(\frac{a+\tau(x,t)}{a+t}(T-t) \right) e^{\lambda_{I+1}-1+j\mu} = 1.$$

Using (2.16) to evaluate the sum gives

$$\lambda_{I+1} = (a+\tau(x,t)) \log \left(\frac{a+T-e^\mu(T-t)}{a+t} \right) + 1.$$

For notational simplicity define

$$\eta(t,\mu) \doteq \log \left(\frac{a+T-e^\mu(T-t)}{a+t} \right), \quad \lambda(x,t;\mu) \doteq (a+\tau(x,t))\eta(t,\mu) + 1. \quad (2.26)$$

Then $\lambda_{I+1} = \lambda(x,t;\mu)$. Choose $C < \infty$ such that $|\ell_k| \le C$ for $0 \le k \le I$. Then

$$\sum_{j=0}^{\infty} \bar{\pi}_j^k e^{\lambda_k - 1 + j\mu - C} \le \sum_{j=0}^{\infty} \pi_j^k(x,t;\Lambda) \le \sum_{j=0}^{\infty} \bar{\pi}_j^k e^{\lambda_k - 1 + j\mu + C}.$$

A calculation of the same sort that gave the display above (2.26) gives

$$(a+k)\eta(t,\mu) + 1 - C \le \lambda_k \le (a+k)\eta(t,\mu) + 1 + C \quad k = 0,\ldots,I.$$

Hence $\lambda_k < \infty$ so long as $\lambda_i < \infty$ for some $i = 0,1,\ldots,I,I+1$, which is true by [6, Corollary 28.2.2]. This completes the proof that for any $(x,t) \in \mathcal{D}_a$ there exists $\Lambda \in \mathbb{R}^{I+3}$ so that $G(x,t;\Lambda) = 0$ and $\pi(x,t;\Lambda)$ is a minimizer of (2.15). $\qquad\square$

The next lemma will focus on the claim that for $(x,t) \in \mathcal{D}_a$, there is only one Λ that satisfies $G(x,t,\Lambda) = 0$, which together with the previous lemma completes the proof of Theorem 2.5.

Lemma 2.8 (Uniqueness of characterization). *For $(x,t) \in \mathcal{D}_a$, there is only one $\Lambda \in \mathbb{R}^{I+3}$ such that $G(x,t,\Lambda) = 0$.*

Proof. Recalling the definition of $\bar{\pi}$ in (2.21), notice that (2.20) is simply

$$\pi_j^k(x,t;\Lambda) = \bar{\pi}_j^k e^{\lambda_k - 1 + j\mu - \ell_{k+j}} \quad \text{for } k = 0,1,\ldots,I+1, j = 0,1,\ldots.$$

As noted previously, for any $(x,t) \in \mathcal{D}_a$ we can assume that each $\{\bar{\pi}_\cdot^k\}$ is a valid probability vector. Thus $\pi_j^k(x,t;\Lambda) \ge 0$ and for each k at least one of $\{\pi_j^k(x,t;\Lambda), j = 0,1,\ldots\}$ is strictly positive. If

$$\alpha_k \doteq \sum_{j=0}^{\infty} \pi_j^k(x,t;\Lambda) \qquad T_k \doteq \sum_{j=0}^{\infty} j\pi_j^k(x,t;\Lambda),$$

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then $\alpha_k > 0$ for each $k = 0, 1, \ldots, I, I+1$ and any $\Lambda \in \mathbb{R}^{I+3}$. Using the particular dependency of $\pi_j^k(x, t; \Lambda)$ on Λ, one can compute

$$
\begin{cases}
\frac{\partial \pi_j^k(x,t;\Lambda)}{\partial \lambda_k} & = & \pi_j^k(x, t; \Lambda) \\
\frac{\partial \pi_j^k(x,t;\Lambda)}{\partial \mu} & = & j\pi_j^k(x, t; \Lambda) \\
\frac{\partial \pi_j^k(x,t;\Lambda)}{\partial \lambda_l} & = & 0 \quad l \neq k.
\end{cases}
$$

It is straightforward to construct a dominating function of the form $\bar{\pi}_j^k \cdot C \cdot D^j$ for suitable constants C and D, and hence by the Lebesgue Dominated Convergence Theorem one can compute $D_\Lambda G(x, t; \Lambda)$ explicitly as

$$
\begin{pmatrix}
-\alpha_0 & \cdots & 0 & -T_0 \\
\vdots & \cdots & \vdots & \vdots \\
0 & \cdots & -\alpha_{I+1} & -T_{I+1} \\
-x_0 T_0 & \cdots & -x_{I+1} T_{I+1} & -\sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j^2 \pi_j^k(x, t; \Lambda)
\end{pmatrix}.
$$

Using elementary row operations to make the matrix upper triangular, we see that $\{-\alpha_k : k = 0, 1, \ldots, I+1\}$ and $\sum_{k=0}^{I+1} \frac{x_k}{\alpha_k} T_k^2 - \sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j^2 \pi_j^k(x, t; \Lambda)$ are the eigenvalues of $D_\Lambda G(x, t; \Lambda)$. We have already observed that $\alpha_k > 0$ for all $k = 0, 1, \ldots, I+1$. Also, for every $k = 0, 1, \ldots, I+1$ the Cauchy-Schwartz inequality implies

$$
\left(\sum_{j=0}^{\infty} j^2 \pi_j^k(x, t; \Lambda) \right) \left(\sum_{j=0}^{\infty} \pi_j^k(x, t; \Lambda) \right) \geq \left(\sum_{j=0}^{\infty} j\pi_j^k(x, t; \Lambda) \right)^2.
$$

It is easy to verify that the necessary condition for equality $[\pi_j^k = j^2 \pi_j^k$ for all $j]$ does not hold. Hence the inequality is strict, and therefore

$$
\left(\sum_{j=0}^{\infty} j^2 \pi_j^k(x, t; \Lambda) \right) > \frac{T_k^2}{\alpha_k}.
$$

Thus $D_\Lambda G(x, t; \Lambda)$ is negative definite for all $\Lambda \in \mathbb{R}^{I+3}$.

Now we can prove the uniqueness of Λ. Suppose there are two different $\Lambda_1, \Lambda_2 \in \mathbb{R}^{I+3}$ such that $G(x, t; \Lambda_i) = 0$, $i = 1, 2$. Define $\Lambda(\epsilon) \doteq \epsilon\Lambda_1 + (1 - \epsilon)\Lambda_2$ and

$$
h(\epsilon) \doteq \langle G(x, t; \Lambda(\epsilon)), \Lambda_1 - \Lambda_2 \rangle.
$$

Then $h'(\epsilon) = (\Lambda_1 - \Lambda_2)^T \cdot D_\Lambda G \cdot (\Lambda_1 - \Lambda_2)$. Since $D_\Lambda G(x, t; \Lambda)$ is always negative definite, $h'(\epsilon) < 0$ for all $0 < \epsilon < 1$. However $h(0) = h(1) = 0$. This contradiction shows that $G(x, t; \Lambda) = 0$ has a unique solution in Λ. $\square$

The next theorem considers differentiability properties of $U(x, t)$. As mentioned previously, we first extend the definition of $U(x, t)$ to a neighborhood of (x, t) in $\mathbb{R}^{I+2} \times \mathbb{R}$, label this extension $\bar{U}(x, t)$, and then show $\bar{U}(x, t)$ is differentiable in the normal Euclidean sense. For our needs (a verification argument) the function $\bar{U}(x, t)$ can be used in lieu of $U(x, t)$.

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Theorem 2.9. *Fix $(x,t) \in \mathcal{D}_a$ and define $U(x,t)$ by (2.15). Then there is an open neighborhood $\mathcal{U}$ of (x,t) and an extension $\bar{U}$ of U from $\mathcal{D}_a \cap \mathcal{U}$ to $\mathcal{U}$ which is differentiable on $\mathcal{U}$.*

Proof. By Theorem 2.5, for any $(x,t) \in \mathcal{D}_a$ there exists Λ so that $G(x,t;\Lambda) = 0$ and $U(x,t) = L\Big(x,t;\Lambda,\pi(x,t;\Lambda)\Big)$. A natural approach to proving smoothness would be to apply the Implicit Function Theorem. There is however a difficulty with this approach, owing to the fact that the non-smooth function $\tau(x,t)$ appears in the constraints involving G_{I+1} and G_{I+2}. To avoid this difficulty we consider an equivalent but less obvious formulation of the constraint.

As discussed above (2.26), if the Lagrange multiplier λ_{I+1} is set to $\lambda(x,t;\mu)$, then the constraint $G_{I+1}(x,t;\Lambda) = 0$ will hold automatically. We will work with the reduced set of multipliers $\tilde{\Lambda} \doteq \{\lambda_0, \ldots, \lambda_I, \mu\}$ and the definition

$$\Lambda\left(x,t;\tilde{\Lambda}\right) \doteq \{\lambda_0, \lambda_1, \ldots, \lambda_I, \lambda(x,t;\mu), \mu\}. \tag{2.27}$$

Setting

$$H(x,t;\tilde{\Lambda}) \doteq L\left(x,t;\Lambda\left(x,t;\tilde{\Lambda}\right); \pi\left(x,t;\Lambda\left(x,t;\tilde{\Lambda}\right)\right)\right) \tag{2.28}$$

gives $U(x,t) = H(x,t;\tilde{\Lambda})$.

To apply the Implicit Function Theorem we must show that there are smooth constraints that characterize $\tilde{\Lambda}$. For $i = 0, \ldots, I$ we use $\tilde{G}_i(x,t;\tilde{\Lambda}) = 0$, where $\tilde{G}_i(x,t;\tilde{\Lambda}) = G_i(x,t;\Lambda)$. These constraints are equivalent since π_j^k does not depend on λ_{I+1} for $k \leq I$. Since $G_{I+1}(x,t;\Lambda) = 0$ holds automatically, we need only define $\tilde{G}_{I+1}$ so that $\tilde{G}_{I+1}(x,t;\tilde{\Lambda}) = 0$ is equivalent to $G_{I+2}(x,t;\Lambda) = 0$. We have

$$G_{I+2}(x,t;\Lambda) = T - t - \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} j\pi_j^k(x,t;\Lambda) - x_{I+1} \sum_{j=0}^{\infty} j\pi_j^{I+1}(x,t;\Lambda),$$

and thus set

$$\tilde{G}_{I+1}(x,t;\tilde{\Lambda}) \doteq G_{I+2}(x,t;\Lambda(x,t;\tilde{\Lambda})). \tag{2.29}$$

Since $\pi_j^k(\cdot)$ does not depend on λ_{I+1} when $k \leq I$ we abuse notation and write the terms of the form $\pi_j^k(x,t;\Lambda(x,t;\tilde{\Lambda}))$ as $\pi_j^k(x,t;\tilde{\Lambda})$. Thus

$$\tilde{G}_{I+1}(x,t;\tilde{\Lambda}) = T - t - \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} j\pi_j^k(x,t;\tilde{\Lambda}) - x_{I+1} \sum_{j=0}^{\infty} j\pi_j^{I+1}(x,t;\Lambda(x,t;\tilde{\Lambda})).$$

Since for $(x,t) \in \mathcal{D}_a$ the value Λ such that $G(x,t;\Lambda) = 0$ exists and is unique, the value $\tilde{\Lambda}$ such that $\tilde{G}(x,t;\tilde{\Lambda}) = 0$ exists and is also unique.

We have that $D_{\tilde{\Lambda}} \tilde{G}(x,t;\tilde{\Lambda})$ equals

$$\begin{pmatrix} -\alpha_0 & \cdots & 0 & -T_0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & -\alpha_I & -T_I \\ -x_0 T_0 & \cdots & -x_I T_I & D_\mu \tilde{G}_{I+1}(x,t;\tilde{\Lambda}) \end{pmatrix},$$

where it is only the last entry that must be identified.

We pause to introduce a convention which will be used in the remainder of the paper. Whenever a differential operator of the form D_x precedes a composed function, the derivative is computed via the chain rule for precisely those arguments where a composed dependence on x is made explicit in the notation. Thus in computing $D_\mu \tilde{G}_{I+1}(x, t; \tilde{\Lambda})$ we use (2.29) and calculations in the last section to get

$$
\begin{aligned}
&D_\mu \tilde{G}_{I+1}(x, t; \tilde{\Lambda}) \\
&= D_\mu G_{I+2}(x, t; \lambda_0, \ldots, \lambda_I, \lambda(x, t; \mu), \mu) \\
&= D_\mu G_{I+2}(x, t; \lambda_0, \ldots, \lambda_I, \lambda_{I+1}, \mu) + D_{\lambda_{I+1}} G_{I+2}(x, t; \lambda_0, \ldots, \lambda_I, \lambda_{I+1}, \mu) \cdot D_\mu \lambda(x, t; \mu) \\
&= -\sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j^2 \pi_j^k(x, t; \Lambda(x, t; \tilde{\Lambda})) - x_{I+1} T_{I+1} \cdot D_\mu \lambda_{I+1}(x, t; \mu).
\end{aligned}
$$

It will be useful to express π^{I+1} in the $\mathcal{Q}^a(s)$ notation. We have

$$
\pi_j^{I+1} = Q_j^{a+\tau(x,t)} \left(\frac{a + \tau(x, t)}{a + t} (T - t) \right) e^{\lambda_{I+1} - 1} e^{j\mu}.
$$

Recall that λ_{I+1} is chosen to make this a probability measure. By (2.16),

$$
e^{\lambda_{I+1} - 1} = \left(1 + \frac{T - t}{a + t}(1 - e^\mu) \right)^{a + \tau(x,t)}.
$$

Hence using a little algebra we can write

$$
\begin{aligned}
&\pi_j^{I+1} \\
&= \left(-\frac{T - t}{a + t} \right)^j \binom{-a - \tau(x, t)}{j} \left(1 + \frac{T - t}{a + t} \right)^{-a - \tau(x,t) - j} e^{j\mu} \left(1 + \frac{T - t}{a + t}(1 - e^\mu) \right)^{a + \tau(x,t)} \\
&= \left(-\frac{e^\mu (T - t)}{a + T - e^\mu (T - t)} \right)^j \binom{-a - \tau(x, t)}{j} \left(\frac{a + T}{a + T - e^\mu (T - t)} \right)^{-a - \tau(x,t) - j} \\
&= Q_j^{a+\tau(x,t)} \left(\frac{e^\mu (T - t)(a + \tau(x, t))}{a + T - e^\mu (T - t)} \right).
\end{aligned}
$$

Again using (2.16)

$$
T_{I+1} = \sum_{j=0}^{\infty} j \pi_j^{I+1}(x, t; \Lambda(x, t; \tilde{\Lambda})) = \frac{e^\mu (a + \tau(x, t))(T - t)}{a + T - e^\mu (T - t)}. \tag{2.30}
$$

Recalling the definition

$$
\lambda(x, t; \mu) = (a + \tau(x, t)) \log \left(\frac{a + T - e^\mu (T - t)}{a + t} \right),
$$

a direct calculation shows $D_\mu \lambda(x, t; \mu) = -T_{I+1}$, and hence

$$
D_\mu \tilde{G}_{I+1}(x, t; \tilde{\Lambda}) = -\sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j^2 \pi_j^k(x, t; \Lambda(x, t; \tilde{\Lambda})) + x_{I+1} T_{I+1}^2. \tag{2.31}
$$

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Since $\alpha_{I+1} = 1$, it follows that the eigenvalues of $D_{\tilde{\Lambda}}\tilde{G}(x,t;\tilde{\Lambda})$ are $-\alpha_0, -\alpha_1, \cdots, -\alpha_I$ and $\sum_{k=0}^{I+1} \frac{x_k}{\alpha_k} T_k^2 - \sum_{k=0}^{I+1} x_k \sum_{j=1}^{\infty} j^2 \pi_j^k(x,t;\Lambda(x,t;\tilde{\Lambda}))$. By the same argument as was used for $D_\Lambda G(x,t;\Lambda)$ these are all negative, and hence $D_{\tilde{\Lambda}}\tilde{G}(x,t;\tilde{\Lambda})$ is invertible.

We next claim that $D_{\tilde{\Lambda}}\tilde{G}$ is smooth in (x,t). One can check that the only potentially difficult component is $D_\mu \tilde{G}_{I+1}(x,t;\tilde{\Lambda})$, and of this the only non-trivial part is

$$x_{I+1} \left(\sum_{j=0}^{\infty} j^2 \pi_j^{I+1}(x,t;\Lambda(x,t;\tilde{\Lambda})) - T_{I+1}^2 \right).$$

Using (2.17) and some algebra shows this term equals

$$-x_{I+1}(a + \tau(x,t)) \cdot \frac{(t-T)e^\mu(a+T)}{[e^\mu(t-T) + a + T]^2}.$$

Although $\tau(x,t)$ is not smooth $x_{I+1}(a + \tau(x,t))$ is always smooth, and thus $D_\mu \tilde{G}$ is smooth in (x,t). Note that the denominator does not vanish since $\eta(t,\mu) > -\infty$.

Therefore $\tilde{G}(\cdot;\cdot)$ is smooth in a neighborhood of $(x,t;\tilde{\Lambda})$. By the Implicit Function Theorem, for any $(x,t) \in \mathcal{D}_a$ there exists a neighborhood $\mathcal{U} \subset \mathbb{R}^{I+1} \times \mathbb{R}$ of (x,t), a neighborhood $\mathcal{V} \subset \mathbb{R}^{I+2}$ of $\tilde{\Lambda}$, and a C^∞ function $g : \mathcal{U} \mapsto \mathcal{V}$, so that $\tilde{\Lambda} = g(x,t)$ and for every $(y,s) \in \mathcal{U}$, $\tilde{G}(y,s;g(y,s)) = 0$. Define

$$\bar{U}(y,s) = H(y,s;g(y,s))$$

Since $g(y,s)$ is smooth in $\mathcal{U}$, $\bar{U} \in C^\infty(\mathcal{U}, \mathbb{R})$, and by Theorem 2.5 $\bar{U}(y,s) = U(y,s)$ for $(y,s) \in \mathcal{U} \cap \mathcal{D}_a$. $\square$

The next theorem expresses the derivatives in terms of the Lagrange multipliers.

Theorem 2.10. *Fix $(x,t) \in D_a$, and let $\tilde{\Lambda}^*$ be the associated Lagrange multiplier. We have*

$$\begin{cases} D_{x_k}\bar{U}(x,t) - D_{x_{k+1}}\bar{U}(x,t) = \lambda_k^* - \lambda_{k+1}^* + \eta^* & k = 0,1,\ldots,I-1 \\ D_{x_I}\bar{U}(x,t) - D_{x_{I+1}}\bar{U}(x,t) = \lambda_I^* - 1 - (a+I)\eta^* \end{cases}$$

and $D_t U(x,t) = \eta^* - \mu^*$.

Proof. Consider any point $(x,t) \in \mathcal{D}_a$ and let $\tilde{\Lambda}^*$ be the associated Lagrange multiplier. By Theorem 2.9 there exists $\mathcal{U} \subset \mathbb{R}^{I+1} \times \mathbb{R}$ a neighborhood of (x,t), $\mathcal{V} \subset \mathbb{R}^{I+2}$ a neighborhood of $\tilde{\Lambda}^*$, and a C^∞ function $\tilde{\Lambda} : \mathcal{U} \mapsto \mathcal{V}$ such that and $\bar{U}(y,s) \doteq H(y,s;\tilde{\Lambda}(y,s))$ satisfies $\bar{U}(y,s) = U(y,s)$ for any $(y,s) \in \mathcal{U} \cap \mathcal{D}_a$.

Keeping in mind the convention regarding differential operators

$$\begin{aligned} D_{x_k}\bar{U}(x,t) &= D_{x_k} H(x,t;\tilde{\Lambda}(x,t)) \\ &= D_{x_k} H(x,t;\tilde{\Lambda}^*) + D_{\tilde{\Lambda}} H(x,t;\tilde{\Lambda}^*) D_{x_k}\tilde{\Lambda}(x,t). \end{aligned}$$

Thus in the first line $H(x,t;\tilde{\Lambda}(x,t))$ is considered as the composed function of (x,t) (which by definition is $\bar{U}(x,t)$), and we take derivatives with respect to two arguments and evaluate at (x,t). In the second line, $D_{x_k} H(x,t;\tilde{\Lambda}^*)$ means $H(x,t;\tilde{\Lambda}^*)$

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is now a function of the independent variables $(x, t, \tilde{\Lambda}^*)$ and we take derivatives with respect to x_k and then evaluate it at $(x, t; \tilde{\Lambda}^*)$. In all calculations, vectors are interpreted as row vectors.

With the notation established, we can proceed. Note that by definition (2.28)

$$D_{\tilde{\Lambda}} H(x, t; \tilde{\Lambda})$$
$$= D_\Lambda L(x, t; \Lambda; \pi) \, D_{\tilde{\Lambda}} \Lambda \left(x, t; \tilde{\Lambda} \right) + D_\pi L(x, t; \Lambda; \pi) \, D_{\tilde{\Lambda}} \pi \left(x, t; \Lambda \left(x, t; \tilde{\Lambda} \right) \right).$$

Since $\Lambda^* = \Lambda(x, t; \tilde{\Lambda}^*)$ and π^* are chosen so that $D_\pi L(x, t; \Lambda^*; \pi^*) = D_\Lambda L(x, t; \Lambda^*; \pi^*) = 0$, we have $D_{\tilde{\Lambda}} H(x, t; \tilde{\Lambda}^*) = 0$. This gives

$$D_{x_k} \bar{U}(x, t) = D_{x_k} H(x, t; \tilde{\Lambda}^*), \qquad k = 0, 1, \ldots, I+1. \tag{2.32}$$

By the same argument, we have $D_t \bar{U}(x, t) = D_t H(x, t; \tilde{\Lambda}^*)$.

Next, we insert the explicit form of $\pi(x, t; \Lambda)$ from (2.20) into (2.19) to get

$$L(x, t; \Lambda, \pi(x, t; \Lambda))$$
$$= \sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} [\lambda_k - 1 + j\mu] \, \pi_j^k(x, t; \Lambda) + \sum_{k=0}^{I+1} \lambda_k x_k \left(1 - \sum_{j=0}^{\infty} \pi_j^k(x, t; \Lambda) \right)$$
$$+ \mu \left(T - t - \sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j \pi_j^k(x, t; \Lambda) \right)$$
$$= -\sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(x, t; \Lambda) - x_{I+1} \sum_{j=0}^{\infty} \pi_j^{I+1}(x, t; \Lambda) + \sum_{k=0}^{I+1} \lambda_k x_k + \mu (T - t).$$

In the definition of $H(x, t; \tilde{\Lambda})$, λ_{I+1} is replaced by $\lambda(x, t; \mu)$ so that automatically $\sum_{j=0}^{\infty} \pi_j^{I+1}(x, t; \Lambda) = 1$. Using $x_{I+1} \lambda(x, t; \mu) - x_{I+1} = x_{I+1}(a + \tau(x, t)) \eta(t, \mu)$ from (2.26) and the definition of $\tau(x, t)$,

$$H(x, t; \tilde{\Lambda}) = \sum_{k=0}^{I} x_k \lambda_k + \mu(T - t) - \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(x, t; \tilde{\Lambda})$$
$$+ x_{I+1}(a + \tau(x, t)) \eta(t, \mu)$$
$$= \sum_{k=0}^{I} x_k \lambda_k + \mu(T - t) - \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(x, t; \tilde{\Lambda})$$
$$+ a \left(1 - \sum_{k=0}^{I} x_k \right) \eta(t, \mu) + \left(t - \sum_{k=0}^{I} k x_k \right) \eta(t, \mu).$$

Recall from the explicit expression (2.20) that for $k = 0, 1, \ldots, I$, $\pi_j^k(x, t; \Lambda)$ does not depend on x or on λ_{I+1}. Hence the x dependence can be omitted in $\pi_j^k(x, t; \tilde{\Lambda})$ in the last display, and we do so from now on.

By (2.32),
$$\bar{U}_{x_k} = \lambda_k^* - 1 - (a + k) \eta^*, \qquad k = 0, 1, \ldots, I,$$

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and $\bar{U}_{x_{I+1}} = 0$, where $\eta^* = \eta(t, \mu^*)$. This implies the first claim of the theorem. Similarly

$$D_t \bar{U}(x, t)$$
$$= D_t H(x, t; \tilde{\Lambda}^*)$$
$$= \eta^* - \mu^* - D_t \left\{ \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(t; \tilde{\Lambda}^*) \right\} + \left\{ a \left(1 - \sum_{k=0}^{I} x_k \right) + \left(t - \sum_{k=0}^{I} k x_k \right) \right\} D_t \eta(t, \mu^*)$$
$$= \eta^* - \mu^* - D_t \left\{ \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(t; \tilde{\Lambda}^*) \right\} + x_{I+1}(a + \tau(x, t)) D_t \eta(t, \mu^*).$$

We now will verify that

$$-D_t \left\{ \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(t; \tilde{\Lambda}^*) \right\} + x_{I+1}(a + \tau(x, t)) D_t \eta(t; \mu^*) = 0, \qquad (2.33)$$

which implies $D_t U(x, t) = \eta^* - \mu^*$. By the explicit formula for $\pi_j^k(t; \tilde{\Lambda})$ in (2.20) and the definition of $\mathcal{Q}_j^a(s)$, for all $k = 0, 1, \ldots, I; j = 0, 1, \ldots$

$$D_t \pi_j^k(t; \tilde{\Lambda}) = j \pi_j^k(t; \tilde{\Lambda}) \frac{a + T}{(t - T)(a + t)} + \frac{(a + k + j)}{a + t} \pi_j^k(t; \tilde{\Lambda}).$$

A suitable dominating function can be found, and thus by the Lebesgue Dominated Convergence Theorem

$$D_t \left\{ \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(t; \tilde{\Lambda}) \right\}$$
$$= \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \left[j \pi_j^k(t; \tilde{\Lambda}) \frac{a + T}{(t - T)(a + t)} + \frac{(a + k + j)}{a + t} \pi_j^k(t; \tilde{\Lambda}) \right].$$

Using that $\pi(x, t; \Lambda^*)$ satisfies the constraint (2.11) and some elementary algebra,

$$D_t \left\{ \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(t; \tilde{\Lambda}^*) \right\}$$
$$= -\frac{x_{I+1} \sum_{j=0}^{\infty} j \pi_j^{I+1}(x, t; \Lambda^*)}{t - T} - \frac{x_{I+1}(a + \tau(x, t))}{a + t}$$

Equation (2.30) and the definition of η in (2.26) give

$$T_{I+1} = \frac{T - t}{a + t}(a + \tau(x, t)) e^{\mu^* - \eta^*}, \qquad (2.34)$$

and hence

$$D_t \left\{ \sum_{k=0}^{I} x_k \sum_{j=0}^{\infty} \pi_j^k(t; \tilde{\Lambda}^*) \right\} = -\frac{a + \tau(x, t)}{a + t} x_{I+1} \left(1 - e^{\mu^* - \eta^*} \right). \qquad (2.35)$$

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The definition of $\eta(t,\mu)$ in (2.26) and $(a+t)e^\eta = a + T - e^\mu(T-t)$ gives

$$D_t\eta(t,\mu) = \frac{e^\mu}{a+T-(T-t)e^\mu} - \frac{1}{a+t} = \frac{e^{\mu-\eta}-1}{a+t}.$$

Finally, combining this with (2.35) gives (2.33). $\qquad\qquad\qquad\qquad\qquad\square$

Our final theorem shows that $\bar{U}$ satisfies the HJB equation (2.14) in the classical sense on $\mathcal{U}$. When combined with a standard verification argument as in [4], this will imply $V(x,t) = \bar{U}(x,t) = U(x,t)$ on $\mathcal{D}_a$.

Theorem 2.11. $\bar{U}$ *satisfies (2.14) on* $\mathcal{D}_a$.

Proof. Having derived various expressions for the derivatives of $\bar{U}$ in terms of the Lagrange multipliers in Theorem 2.10, to show that $\bar{U}(x,t)$ satisfies the PDE (2.14) it remains to show

$$e^{-\mu^*+\eta^*} = \sum_{k=0}^{I-1} x_k\frac{a+k}{a+t}e^{\lambda_k^*-\lambda_{k+1}^*+\eta^*} + x_I\frac{a+I}{a+t}e^{\lambda_I^*-1-(a+I)\eta^*} + x_{I+1}\frac{a+\tau(x,t)}{a+t}. \quad (2.36)$$

Recall from (2.20) that for $k = 0,1,\ldots,I$ and $j \geq 0$

$$\pi_j^{*k} = \mathcal{Q}_j^{a+k}\left(\frac{a+k}{a+t}(T-t)\right)e^{\lambda_k^*-1+j\mu^*-\ell_{k+j}}.$$

Using the definition of $\mathcal{Q}_j^{a+k}$, for $k = 0,1,\ldots,I-1$

$$\pi_j^{*k+1} = \frac{(j+1)(a+t)}{(a+k)(T-t)}\pi_{j+1}^{*k}e^{\lambda_{k+1}^*-\lambda_k^*}e^{-\mu^*}.$$

Now sum both sides from $j = 0$ to ∞ and use the fact that $\sum_{j=0}^\infty \pi_j^{*k} = 1$ to get

$$e^{\lambda_k^*-\lambda_{k+1}^*} = e^{-\mu^*}\cdot\frac{(a+t)\sum_{j=1}^\infty j\pi_j^{*k}}{(T-t)(a+k)}. \quad \text{for } k = 0,1,\ldots,I-1. \quad (2.37)$$

Inserting (2.37) into (2.36), a little algebra shows that satisfaction of the PDE is equivalent to

$$(T-t) = \sum_{k=0}^{I-1}x_k\sum_{j=0}^\infty j\pi_j^{*k} + x_I\frac{a+I}{a+t}(T-t)e^{\lambda_I^*-1-(a+I+1)\eta^*+\mu^*}$$

$$+x_{I+1}\frac{a+\tau(x,t)}{a+t}(T-t)e^{-\eta^*+\mu^*}. \quad (2.38)$$

Since $\pi_j^{*I} = \mathcal{Q}_j^{a+I}\left(\frac{a+I}{a+t}(T-t)\right)e^{\lambda_I^*-1+j\mu^*}$ for $j \geq 1$, by (2.16)

$$\sum_{j=1}^\infty j\pi_j^{*I} = \sum_{j=1}^\infty j\mathcal{Q}_j^{a+I}\left(\frac{a+I}{a+t}(T-t)\right)e^{\lambda_I^*-1+j\mu^*}$$

$$= \frac{T-t}{a+t}(a+I)e^{\lambda_I^*-1-\eta^*(a+I+1)+\mu^*} \quad (2.39)$$

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where the last equality uses the definition of η in (2.26).

Inserting (2.34) and (2.39) into $\sum_{k=0}^{I+1} x_k \sum_{j=0}^{\infty} j\pi_j^k = T - t$, we then have

$$
\begin{aligned}
(T - t) \quad = \quad & \sum_{k=0}^{I-1} x_k \sum_{j=0}^{\infty} j\pi_j^{*k} + x_I \frac{a + I}{a + t}(T - t)e^{\lambda_I^* - 1 - \eta^*(a + I + 1) + \mu^*} \\
& + x_{I+1} \frac{a + \tau(x, t)}{a + t}(T - t)e^{-\eta^* + \mu^*}.
\end{aligned}
$$

We have thus verified (2.38), which completes the proof that $\bar{U}$ satisfies (2.14). $\quad\square$

2.4 Minimizing Trajectories

The minimizing trajectories associated with the calculus of variations problem have important qualitative and computational uses. Perhaps the most important is that they identify the most likely way a rare event will occur [5].

Identification of the minimizing trajectories in the MB case was done in [2] using the Euler-Lagrange equations, a system of nonlinear ordinary differential equations. The solutions to these equation are called extremals, and in general being an extremal is neither a necessary or sufficient condition for minimality. In [2], a direct but detailed argument using Lagrange multiplier techniques was used to show the extremals were indeed minimizers. Here we take a different tack. We start in [2] with a two-parameter family of solutions to the Euler-Lagrange equations that identify the extremals. The two parameters are themselves characterized as the solution to a pair of nonlinear algebraic equations. These parameters and the form of the extremals suggest values for the Lagrange multipliers in the explicit representation (2.15), and indeed it is shown that characterizing equations $G(x, t; \Lambda) = 0$ for the unique Lagrange multipliers are satisfied. Having identified the minimal cost, all that remains is the show that the cost along the extremal is the same as this minimal value. This is done by explicitly evaluating an integral.

The main goal of this section is to argue that the extremals are minimizers and exhibit the relation between the two parameters used to identify the extremals and the Lagrange multipliers used in the formula for the minimal cost. Not all details will be given, and to simplify the presentation only the initial condition with $t = 0$ and all cells empty is considered. The statement of the case of general initial conditions is exactly analogous to Theorem 2.8 in [2], and any details that are omitted are similar to ones appearing in [2]. In addition, we present only the case $a \in (0, \infty)$. In this case it is simpler to work with an infinite dimensional version of the extremals. This is analogous to what is called *exponential* case in [2]. The arguments for the cases $a = \infty$ and $a < 0$ are analogous to that of $a \in (0, \infty)$. As noted above, the case $a = \infty$ has already been considered in [2]. In the case $a < 0$, the extremals satisfy $\varphi_I = 0$. Because of this the arguments are somewhat simpler than in the case of $a > 0$, and it is analogous to the *polynomial* case of [2].

In the case of the empty initial condition the extremal can be identified as follows. Let $y \in \mathcal{S}_I$ be given. Then a family of solutions to the Euler-Lagrange equations for

the problem of minimizing the cost subject to this terminal condition are

$$\varphi_0(t) = C\mathcal{Q}_0^a(\rho t) + \sum_{k=0}^{I} (y_k - C\mathcal{Q}_k^a(\rho T)) \left(1 - \frac{t}{T}\right)^k$$

$$\varphi_i(t) = \frac{t^i}{i!}(-1)^i \varphi_0^{(i)}(t), \quad 1 \le i \le I,$$

$$\varphi_{I+}(t) = 1 - \sum_{i=0}^{I} \varphi_i(t).$$

This is exactly analogous to the form found in [2] for the special case of $a = \infty$, with the Poisson distribution $\mathcal{P}(t)$ in that case replaced by the family of probability distributions $\mathcal{Q}^a(t)$. It is useful to extend the definition of $\varphi_i(t)$ to $\varphi_i(t) = C\mathcal{Q}_i^a(\rho t)$ for $i > I$, while maintaining the distinction between $\varphi_{I+}(t)$ and $\varphi_I(t)$.

The parameters $\rho > 0$ and $C \ge 0$ are chosen so that the measure corresponding to $\varphi_i(T)$ is a probability measure, and moreover one for which the number of balls per cell at time T equals T. Specifically, ρ is chosen so that

$$\frac{\rho T - \sum_{i=0}^{I} i\mathcal{Q}_i^a(\rho T)}{1 - \sum_{i=0}^{I} \mathcal{Q}_i^a(\rho T)} = \frac{T - \sum_{i=0}^{I} iy_i}{1 - \sum_{i=0}^{I} y_i}$$

holds, and then

$$C \doteq \frac{1 - \sum_{i=0}^{I} y_i}{1 - \sum_{i=0}^{I} \mathcal{Q}_i^a(\rho T)} = \frac{T - \sum_{i=0}^{I} iy_i}{\rho T - \sum_{i=0}^{I} i\mathcal{Q}_i^a(\rho T)}.$$

Solutions to these equations exist for $\rho \in (0, \infty)$ and $C \in [0, \infty)$, and are unique.

To show that this is indeed a minimizing trajectory we relate the constants ρ and C to the Lagrange multipliers appearing in the finite dimensional representation (2.15). Recall that the minimizer to this problem takes the form

$$\pi_j^0(1, 0; \Lambda) = \mathcal{Q}_j^a(T)e^{\lambda_0 - 1 + j\mu - \ell_j},$$

with $\ell_j = 0$ if $j > I$. Using the form of the minimizing trajectory, at time $t = T$

$$\varphi_j(T) = y_j, 0 \le j \le I,$$
$$\varphi_j(T) = C\mathcal{Q}_j^a(\rho T), I < j.$$

Thus for all $j > I$ we will want

$$\mathcal{Q}_j^a(T)e^{\lambda_0 - 1 + j\mu} = C\mathcal{Q}_j^a(\rho T).$$

Since

$$\frac{\mathcal{Q}_j^a(\rho T)}{\mathcal{Q}_j^a(T)} = \frac{\left(\frac{-\rho T}{a+\rho T}\right)^j \left(\frac{a+\rho T}{a}\right)^{-a}}{\left(\frac{-T}{a+T}\right)^j \left(\frac{a+T}{a}\right)^{-a}} = \rho^j \left(\frac{a+T}{a+\rho T}\right)^j \left(\frac{a+T}{a+\rho T}\right)^a,$$

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this suggests

$$\mu = \log \rho + \log \left(\frac{a+T}{a+\rho T} \right)$$

$$\lambda_0 - 1 = a \log \left(\frac{a+T}{a+\rho T} \right) + \log C$$

and

$$\ell_k = -\log y_k + \log \mathcal{Q}_k^a(\rho T) + \log C$$

when $k \leq I$. Hence the minimizing trajectory for a problem with a finite terminal cost ℓ will terminate at a point y with each $y_k > 0$, an assumption we make for the rest of this section. The argument when one or more $y_k = 0$ can be handled by a limiting argument. We remark in passing that similar considerations allow one to explicitly identify the Lagrange multipliers for all initial conditions (x, t) that lie on the extremal in terms of C, ρ, and the values y_k.

To show that $\pi^0(1, 0; \Lambda)$ is the minimizing probability measure in (2.15) the constraints (2.11) must be demonstrated. One constraint is that $\pi_j^0(1, 0; \Lambda)$ be a probability measure. Since the definitions of the Lagrange multipliers enforce $\pi_j^0(1, 0; \Lambda) = \varphi_j(T)$, this follows from $\sum_{i=0}^{I} y_i = 1 - C + C \sum_{i=0}^{I} \mathcal{Q}_i^a(\rho T)$ and

$$\sum_{j=0}^{\infty} \varphi_j(T) = \sum_{j=0}^{I} y_j + \sum_{j=I+1}^{\infty} C \mathcal{Q}_j^a(\rho T) = 1 - C + C = 1.$$

The only other constraint to check is the conservation condition:

$$
\begin{aligned}
\sum_{j=0}^{\infty} j \pi_j^0(1, 0; \Lambda) &= \sum_{j=0}^{\infty} j \mathcal{Q}_j^a(T) e^{\lambda_0 - 1 + j\mu - \ell_j} \\
&= \sum_{j=0}^{\infty} j C \mathcal{Q}_j^a(\rho T) + \sum_{j=0}^{I} j y_j - \sum_{j=0}^{I} j C \mathcal{Q}_j^a(\rho T) \\
&= C \rho T - \sum_{j=0}^{I} j C \mathcal{Q}_j^a(\rho T) + \sum_{j=0}^{I} j y_j \\
&= T - \sum_{i=0}^{I} i y_i + \sum_{j=0}^{I} j y_j \\
&= T,
\end{aligned}
$$

where the equations characterizing C and ρ are used for the fourth equality.

We have identified the optimal measure for the terminal cost ℓ. To complete the argument that φ is a minimizer we need only show that the cost along this trajectory

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equals

$$R\left(\pi^0(1,0;\Lambda)\,\|\,\mathcal{Q}^a(T)\right)$$

$$= \sum_{j=0}^{\infty} \pi_j^0(1,0;\Lambda)\log\left(\frac{\pi_j^0(1,0;\Lambda)}{\mathcal{Q}_j^a(T)}\right)$$

$$= \sum_{j=0}^{\infty} \mathcal{Q}_j^a(T)e^{\lambda_0-1+j\mu-\ell_j}\log\left(e^{\lambda_0-1+j\mu-\ell_j}\right)$$

$$= \sum_{j=0}^{I} y_j\log\left(\frac{y_j}{\mathcal{Q}_j^a(T)}\right) + \sum_{j=I+1}^{\infty} C\mathcal{Q}_j^a(\rho T)\log\left(\frac{C\mathcal{Q}_j^a(\rho T)}{\mathcal{Q}_j^a(T)}\right)$$

$$= \sum_{j=0}^{\infty} \varphi_j(T)\log\left(\frac{\varphi_j(T)}{\mathcal{Q}_j^a(T)}\right)$$

$$= \sum_{j=0}^{\infty} \varphi_j(T)\log\left(\varphi_j(T)\right) - \sum_{j=0}^{\infty} \varphi_j(T)\log\left(\mathcal{Q}_j^a(T)\right)$$

$$= \sum_{j=0}^{\infty} \varphi_j(T)\log\left(\varphi_j(T)\right)$$

$$\quad - \sum_{j=0}^{\infty} \varphi_j(T)\left[\log\left(\frac{T}{a}\right)^j + \log\left(\frac{\prod_{k=0}^{j-1}(a+k)}{j!}\right) + \log\left(\frac{a+T}{a}\right)^{-a-j}\right]$$

$$= \sum_{j=0}^{\infty} \varphi_j(T)\log\left(\varphi_j(T)\right) - T\log\left(\frac{T}{a}\right) + (a+T)\log\left(\frac{a+T}{a}\right)$$

$$\quad - \sum_{j=0}^{\infty} \varphi_j(T)\log\left(\frac{\prod_{k=0}^{j-1}(a+k)}{j!}\right).$$

The notion of a completely monotone function is useful here. Although it is clear from the construction of the Lagrange multipliers that $\varphi_j(T)$ is a probability measure, the same cannot be said yet for $\varphi_j(t)$. A monotone function γ is completely monotone on $[0, T]$ if it is infinitely differentiable on $[0, T]$ and if for all $t \in [0, T]$ and $i \geq 0$

$$(-1)^i\gamma^{(i)}(t) \geq 0.$$

The same argument as [2, p. 2794] shows that $\varphi_0(t)$ is completely monotone on $[0, T]$, and hence for all $t \in [0, T]$ and $i \geq 0$, $\varphi_i(t) \geq 0$. From the explicit formula for $\varphi_0(t)$ we actually have $(-1)^i\varphi_0^{(i)}(t) > 0$ for $t \in (0, T)$. It is also easy to show that for all $t \in [0, T]$ and $i \geq 0$, $\varphi_i(t)$ can be interpreted as the ith term in the Taylor series expansion of $\varphi_0(0)$ about t, and so for each t $\{\varphi_i(t), i = 0, 1, \ldots\}$ is a probability measure on $\{0, 1, \ldots\}$.

To evaluate the cost

$$\int_0^T R\left(\theta\,\|\,\rho(t, \varphi(t))\right)dt,$$

it is convenient to work, as in [2], with the cumulative occupancy functions

$$\psi_j(t) = \sum_{i=0}^{j} \varphi_i(t).$$

The dynamics then take the form $\psi_j^{(1)}(t) = -\theta_j(t)$, and so with the convention $\psi_{-1}(t) = 0$ the cost can be expressed

$$\int_0^T \left[\sum_{i=0}^{I} -\psi_i^{(1)}(t) \log\left(\frac{-\psi_i^{(1)}(t)}{\frac{a+i}{a+t}\left(\psi_i(t) - \psi_{i-1}(t)\right)} \right) \right.$$
$$\left. + \left(1 + \psi_0^{(1)}(t) + \cdots + \psi_I^{(1)}(t)\right) \log\left(\frac{1 + \psi_0^{(1)}(t) + \cdots + \psi_I^{(1)}(t)}{\frac{a}{a+t}\left(1 - \psi_I(t)\right) + \frac{1}{a+t}\sum_{k=I+1}^{\infty} k\left(\psi_k(t) - \psi_{k-1}(t)\right)} \right) \right] dt.$$

We have

$$-\psi_i^{(1)}(t) = -\sum_{k=0}^{i} \varphi_i^{(1)}(t)$$

$$= -\sum_{k=0}^{i} \frac{(-t)^k}{k!}\psi_0^{(k+1)}(t) + \sum_{k=1}^{i} \frac{(-t)^{k-1}}{(k-1)!}\psi_0^{(k)}(t)$$

$$= -\frac{(-t)^i}{i!}\psi_0^{(i+1)}(t),$$

and so

$$\frac{-\psi_i^{(1)}(t)}{\varphi_i(t)} = \frac{-\frac{(-t)^i}{i!}\psi_0^{(i+1)}(t)}{\frac{(-t)^i}{i!}\psi_0^{(i)}(t)} = \frac{-\psi_0^{(i+1)}(t)}{\psi_0^{(i)}(t)}.$$

Since for $i > I$

$$\frac{-\psi_i^{(1)}(t)}{\frac{a+i}{a+t}\left(\psi_i(t) - \psi_{i-1}(t)\right)} = \frac{a+t}{a+i} \cdot \frac{-\varphi_0^{(i+1)}(t)}{\varphi_0^{(i)}(t)}$$

$$= \frac{a+t}{a+i} \cdot \frac{\mathcal{Q}_{i+1}^a(\rho t)(i+1)}{\mathcal{Q}_i^a(\rho t)t}$$

$$= \frac{a+t}{a+i} \cdot \frac{(-\rho t)(-a-i)}{(a+\rho t)t}$$

$$= \frac{(a+t)\rho}{a+\rho t}.$$

and

$$\frac{1 + \psi_0^{(1)}(t) + \cdots + \psi_I^{(1)}(t)}{\frac{a}{a+t}\left(1 - \psi_I(t)\right) + \frac{1}{a+t}\sum_{k=I+1}^{\infty} k\left(\psi_k(t) - \psi_{k-1}(t)\right)}$$

$$= \frac{\sum_{i=I+1}^{\infty} -\frac{(-t)^i}{i!}\varphi_0^{(i+1)}(t)}{\frac{a}{a+t}\left(\sum_{i=I+1}^{\infty} \frac{(-t)^i}{i!}\varphi_0^{(i)}(t)\right) + \frac{1}{a+t}\sum_{i=I+1}^{\infty} i\frac{(-t)^i}{i!}\varphi_0^{(i)}(t)}$$

$$= \frac{\sum_{i=I+1}^{\infty} -\frac{(-t)^i}{i!}\varphi_0^{(i+1)}(t)}{\frac{a+i}{a+t}\left(\sum_{i=I+1}^{\infty} \frac{(-t)^i}{i!}\varphi_0^{(i)}(t)\right)},$$

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we can write the integral as

$$\int_0^T \sum_{i=0}^\infty -\psi_i^{(1)}(t) \log\left(\frac{-\psi_i^{(1)}(t)}{\frac{a+i}{a+t}\left(\psi_i(t) - \psi_{i-1}(t)\right)}\right) dt.$$

At several places below we will need the existence of a dominating function to justify the interchange of summation and integration. A suitable function can be found using the same calculations as those used to establish (2.16). Also, this dominating function will work only on closed subintervals of $(0, T)$, and so a careful argument will first evaluate the integral on $[\varepsilon, T - \varepsilon]$ and then use monotone convergence to take the limit $\varepsilon \downarrow 0$.

We evaluate the integral by the following calculation, each line of which is explained below:

$$\int_0^T \sum_{i=0}^\infty -\psi_i^{(1)}(t) \log\left(\frac{a+t}{a+i}\frac{-\psi_0^{(i+1)}(t)}{\psi_0^{(i)}(t)}\right) dt$$

$$= \int_0^T \sum_{i=0}^\infty \left(-\psi_i^{(1)}(t) \log\left|\psi_0^{(i+1)}(t)\right| + \psi_i^{(1)}(t) \log\left|\psi_0^{(i)}(t)\right| - \psi_i^{(1)}(t) \log\left(a+i\right)\right) dt$$

$$+ \int_0^T \left(\sum_{i=0}^\infty -\psi_i^{(1)}(t)\right) \log\left(a+t\right) dt$$

$$= \int_0^T \sum_{i=0}^\infty \left[\left(\psi_i^{(1)}(t) - \psi_{i-1}^{(1)}(t)\right) \log\left|\psi_0^{(i)}(t)\right| - \psi_i^{(1)}(t) \log\left(a+i\right)\right] dt$$

$$+ \int_0^T \log\left(a+t\right) dt$$

$$= \sum_{i=0}^\infty \left(\left[\varphi_i(t) \log\left|\psi_0^{(i)}(t)\right|\right]_0^T + \int_0^T \left(\sum_{i=0}^\infty -\psi_i^{(1)}(t)\right) dt + \sum_{i=0}^\infty \left[\psi_i(t)\right]_0^T \log(a+i)\right)$$

$$+ (a+T) \log(a+T) - (a+T) - a \log a + a$$

$$= \sum_{i=0}^\infty \left(\left[\varphi_i(T) \log\left|\varphi_0^{(i)}(T)\right| - \varphi_i(0) \log\left|\varphi_0^{(i)}(0)\right|\right]_0^T + \log(a+i) \left(\sum_{k=0}^i \varphi_i(T) - \sum_{k=0}^i \varphi_i(0)\right)\right)$$

$$+ T + (a+T) \log(a+T) - (a+T) - a \log a + a$$

$$= \sum_{i=0}^\infty \left(\varphi_i(T) \log\left(\varphi_i(T) i!/T^i\right) - \log(a+i) \left(\sum_{k=i+1}^\infty \varphi_k(T)\right)\right)$$

$$+ (a+T) \log(a+T) - a \log a$$

$$= \sum_{i=0}^\infty \left(\varphi_i(T) \log \varphi_i(T) + \sum_{i=0}^\infty \varphi_i(T) \log\left(i!\right) - \varphi_i(T) \log\left(\prod_{k=0}^{i-1}(a+k)\right)\right)$$

$$- T \log T + (a+T) \log(a+T) - a \log a$$

$$= \sum_{i=0}^\infty \varphi_i(T) \log \varphi_i(T) + \sum_{i=0}^\infty \varphi_i(T) \log\left(\frac{i!}{\prod_{k=0}^{i-1}(a+k)}\right)$$

$$- T \log T + (a+T) \log(a+T) - a \log a.$$

The first line separates the $a + t$ term in the logarithm. The second equality uses the convention $\psi_{-1}(t) = 0$ and that $\sum_{i=0}^{\infty} -\psi_i^{(1)}(t) = 1$ (all balls go into some cell). The third line uses integration by parts and the fourth uses the definition of the cumulative occupancies. The fifth line uses the definition of φ_i in terms of derivatives of φ_0, and the sixth equality uses summation by parts. Since

$$-T \log\left(\frac{T}{a}\right) + (a + T) \log\left(\frac{a + T}{a}\right) = -T \log T + (a + T) \log(a + T) - a \log a,$$

the cost along this trajectory equals the minimum, and the argument is complete. □

3   Allocation Models with Balls of Different Color

In this section we extend the techniques to the case of allocation models where the balls are of more than one type. To keep the notation simple, we actually consider just two colors, the extension to the more general case being straightforward. Another simplification is that we consider only MB statistics. The interested reader can combine the methods from this section and the last to treat more general statistical models.

3.1   Coloration Process

We construct an allocation model with colored balls as follows. Balls are thrown into one of n cells sequentially. The throwing process is modeled by a collection of independent and identically distributed (iid) random variables, each uniformly distributed on the set $\{1, \ldots, n\}$, with each value of the set corresponding to an cell. There is also a coloration process. At each discrete time a ball is assigned color $Y_l^n \in \{1, 2\}$, and then placed into the cell determined by the throwing process. These two processes are independent.

The occupancy process in this case is defined as follows. The natural state space is

$$S_{I,J} \doteq \left\{ x \in \mathbb{R}^{I+2} \times \mathbb{R}^{J+2} : x_{i,j} \geq 0, 0 \leq i \leq I + 1, 0 \leq j \leq J + 1, \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} = 1 \right\}.$$

If $i \in \{0, \ldots, I\}$ and $j \in \{0, \ldots, J\}$, then $X_{i,j}^n(l/n)$ is the fraction of cells containing exactly i color-1 and j color-2 balls when l balls have been thrown. In an analogous fashion $X_{I+,j}^n(l/n), X_{i,J+}^n(l/n)$ and $X_{I+,J+}^n(l/n)$ are defined. By definition $X_{0,0}^n(0) \doteq 1$, and $X_{i,j}^n(0) \doteq 0$ for all other values of (i, j).

To describe the large deviation asymptotics of these allocation processes we must specify those of the coloration processes. Cumulative coloration processes $\{r^n, n \in \mathbb{N}\}$ are defined for $t = l/n$ by

$$r_1^n(l/n) \doteq \frac{1}{n} \sum_{k=1}^{l} 1_{\{Y_k^n = 1\}}, \quad r_2^n(l/n) \doteq \frac{1}{n} \sum_{k=1}^{l} 1_{\{Y_k^n = 2\}}.$$

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We will assume that these processes satisfy a large deviations principle with a rate function of the form $J(\phi) = \int_0^T c(\dot{\phi}(s))ds$. Thus $c(\dot{\phi}(s))$ is a measure of the local (in time) log likelihood that a fraction $\dot{\phi}_i(s)$ of the balls are color i. A mild technical assumption that is needed to prove a large deviations result for the occupancy process is that $c(a) = 0$ for some point a with $a_i > 0, i = 1, 2$. Since c is a rate function, there is at least one probability vector a at which $c(a) = 0$. The assumption that this occurs at a point where both components are positive is very mild, and means simply that the LLN limit cannot concentrate exclusively on one color.

Examples of coloration processes which satisfy these properties are *deterministic, iid* and *Markovian.* In the iid case colors are selected by an iid sequence of random variables. In the Markov case the color is chosen by a finite state ergodic Markov chain. The so-called deterministic case seeks to achieve a deterministic fraction a_k of color k, with $a_k \in (0, 1)$. This can be done as follows. If N_{l-1}^k balls of color k have been thrown in the first $l - 1$ throws (with $N_{l-1}^1 + N_{l-1}^2 = l - 1$), and if $N_{l-1}^1/n \le a_1 l/n$, then we color the lth ball 1, and otherwise color it 2.

The specific form for c in all these cases is spelled out in [3]. For reasons to be explained below, the focus in this paper will be on the iid and deterministic cases, where $c(\rho)$ equals $R(\rho \| a)$ and $\infty \cdot 1_{\{a\}^c}(\rho)$, repsectively. Under a suitable restriction needed to ensure convexity that is also described below, the same methods can be applied to the Markovian case as well.

3.2 Variational Problem and PDE

For this problem the feasible domain is

$$
\mathcal{D} \doteq \left\{ (x, t) \in S_{I,J} \times [0, T) : \sum_{i=0}^{I} x_{i,J+1} + \sum_{j=0}^{J} x_{I+1,j} + x_{I+1,J+1} > 0 \text{ and} \right.
$$

$$
\left. t > \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} \text{ or } \sum_{i=0}^{I} x_{i,J+1} + \sum_{j=0}^{J} x_{I+1,j} + x_{I+1,J+1} = 0 \text{ and } t = \sum_{i=0}^{I} \sum_{j=0}^{J} x_{i,j} \right\}.
$$

We next describe the large deviations variational problem. Recall that $S_{I,J}$ are the set of all probability measures on $\{0, I + 1\} \times \{0, J + 1\}$, which can also be interpreted as real $(I + 2) \times (J + 2)$ matrices. For $\alpha \in S_{I,J}$ define the linear maps

$$
M_{i,j}^1[\alpha] = \alpha_{i-1,j} 1_{\{i \ge 1\}} - \alpha_{i,j} 1_{\{i \le I\}}, M_{i,j}^2[\alpha] = \alpha_{i,j-1} 1_{\{j \ge 1\}} - \alpha_{i,j} 1_{\{j \le J\}}.
$$

The rate function for the coloration over an interval $[t, T]$ is assumed to be of the form $\int_t^T c(\rho)ds$, where $\rho(s) = (\rho_1(s), \rho_2(s))$ are the colored fractions at time s. The local (in time) and total coloration fractions satisfy $q_k = \int_t^T \rho_k(s)ds/[T - t]$, and for a trajectory of the form $(\rho_1(s), \rho_2(s)) = (q_1, q_2)$, the cost is of course $[T - t]c(q)$. The rate function for the occupancy process on path space is then

$$
\mathcal{I}(x, t; \varphi) = \inf_{\theta, \rho} \int_t^T \left[\rho_1 R(\theta^1 \| \varphi) + \rho_2 R(\theta^2 \| \varphi) + c(\rho) \right] ds,
$$

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where the infimum is over all θ, ρ such that

$$\varphi(u) - \varphi(t) = \int_t^u \left(\rho_1 M^1[\theta^1] + \rho_2 M^2[\theta^2]\right) ds.$$

For a terminal cost F we consider

$$V(x,t) = \inf_{\varphi \in C([t,T], S_{I,J}), \varphi(t) = x} \left\{\mathcal{I}(x,t;\varphi) + F(\varphi(T))\right\}.$$

Then V should be a weak-sense solution to

$$W_t + H(W_x, x, t) = 0$$

and the terminal condition, where

$$H(p,x,t) = \inf_{\rho,\theta^1,\theta^2} \left[\langle p, \rho_1 M^1[\theta^1] + \rho_2 M^2[\theta^2]\rangle + \rho_1 R(\theta^1 \,\|\, x) + \rho_2 R(\theta^2 \,\|\, x) + c(\rho)\right].$$

If $b(\gamma)$ is the Legendre transform $b(\gamma) = \sup_\rho \left[\langle \gamma, \rho\rangle - c(\rho)\right]$, then we can also write

$$
\begin{aligned}
H(p,x,t) &= -\sup_\rho \left[-\sum_{m=1,2} \rho_m \left(\inf_{\theta^m} \left[\langle p, M^m[\theta^m]\rangle + R(\theta^m \,\|\, x)\right]\right) - c(\rho) \right] \\
&= -b\left(-\inf_{\theta^1}\left[\langle p, M^1[\theta^1]\rangle + R(\theta^1 \,\|\, x)\right], -\inf_{\theta^2}\left[\langle p, M^2[\theta^2]\rangle + R(\theta^2 \,\|\, x)\right]\right).
\end{aligned}
$$

The variational formula for exponential integrals in terms of relative entropy [1, Proposition 1.4.2] asserts that

$$
\begin{aligned}
\inf_{\theta^1}\left[\langle p, M^1[\theta^1]\rangle + R(\theta^1 \,\|\, x)\right] &= \inf_{\theta^1}\left[\sum_{i,j,i\geq 1} p_{i,j}\theta^1_{i-1,j} - \sum_{i,j,i\leq I} p_{i,j}\theta^1_{i,j} + R(\theta^1 \,\|\, x)\right] \\
&= \inf_{\theta^1}\left[\sum_{i,j,i\leq I}(p_{i+1,j} - p_{i,j})\theta^1_{i,j} + R(\theta^1 \,\|\, x)\right] \\
&= -\log\left(\sum_{i,j,i\leq I} e^{-(p_{i+1,j}-p_{i,j})}x_{i,j} + \sum_{i,j,i=I+1} x_{i,j}\right).
\end{aligned}
$$

Using the analogous formula for $m = 2$, one obtains

$$
\begin{aligned}
H(p,x,t) = -b\Bigg[&\log\left(\sum_{i,j,i\leq I} e^{-(p_{i+1,j}-p_{i,j})}x_{i,j} + \sum_{i,j,i=I+1} x_{i,j}\right), \\
&\log\left(\sum_{i,j,j\leq J} e^{-(p_{i,j+1}-p_{i,j})}x_{i,j} + \sum_{i,j,j=J+1} x_{i,j}\right)\Bigg]. \qquad (3.1)
\end{aligned}
$$

3.3 Explicit Solution

Let $\pi_{i,j}(r_1, r_2)$ denote the probability of throwing r_m additional balls of color m, $m = 1, 2$, into cells of category (i, j), and let $q = (q_1, q_2)$ be the fraction of balls of colors $(1, 2)$. For $x \in S_{I,J}$, we say that $(\pi, q) \in \mathcal{F}(x, t; y, T)$ if for all i, j

$$x_{i,j} \sum_{r_1, r_2 = 0}^{\infty} \pi_{i,j}(r_1, r_2) = x_{i,j}, \quad \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} \sum_{r_1, r_2 = 0}^{\infty} r_m \pi_{i,j}(r_1, r_2) = q_m(T - t)$$

for $m = 1, 2$ and

$$\begin{aligned}
y_{k,l} &= \sum_{i=0}^{k} \sum_{j=0}^{l} x_{i,j} \pi_{i,j}(k - i, l - j) \\
y_{I+1,l} &= \sum_{r=0}^{\infty} \sum_{i=0}^{I+1} \sum_{j=0}^{l} x_{i,j} \pi_{i,j}(I + 1 - i + r, l - j) \\
y_{k,J+1} &= \sum_{r=0}^{\infty} \sum_{i=0}^{k} \sum_{j=0}^{J+1} x_{i,j} \pi_{i,j}(k - i, J + 1 - j + r) \\
y_{l+1,J+1} &= \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} \pi_{i,j}(I + 1 - i + s, J + 1 - j + r).
\end{aligned}$$

We also denote y by $x \times \pi$. If the coloration turns out to be (q_1, q_2), then there are $q_m(T - t)n$ balls of color m thrown, and the law of large numbers limit for the empirical fraction of cells of category (i, j) is $\mathcal{P}_i(q_1(T - t))\mathcal{P}_j(q_2(T - t))$.

The same sort of argument as in Section 2.2 then suggests that the explicit form for the solution to the variational problem should be

$$\min_{(\pi, q) \in \mathcal{F}(x, t; y, T)} \left\{ \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} R\left(\pi_{i,j} \| \mathcal{P}(q_1(T - t)) \times \mathcal{P}(q_2(T - t))\right) + (T - t)c(q) \right\}.$$

However, an interesting feature of the case with color is that the quantity being minimized in this formula is not always convex. In a previous paper [3], a useful assumption that guaranteed the convexity of the large deviation rate on path space was that $c(\rho) + h(\rho)$ be convex, where $h(\rho)$ is the entropy function $h(\rho) \doteq -\rho_1 \log \rho_1 - \rho_2 \log \rho_2$. We will show that this same condition, not surprisingly, gives convexity here as well. Let a be the point with $a_i > 0, i = 1, 2$, for which $c(a) = 0$. Then we

can write, under the constraint that relates $\pi_{i,j}$ and q_m,

$$
\sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} R\left(\pi_{i,j} \,\|\, \mathcal{P}(q_1(T-t)) \times \mathcal{P}(q_2(T-t))\right) + (T-t)c(q)
$$

$$
= \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \pi_{i,j}(k,l) \log\left(\frac{\pi_{i,j}(k,l)}{\mathcal{P}_k(a_1(T-t))\mathcal{P}_l(a_2(T-t))} \right)
$$

$$
+ \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \pi_{i,j}(k,l) \left[(T-t)c(q) - \log\left(\left[\frac{q_1}{a_1}\right]^k \left[\frac{q_2}{a_2}\right]^l \right) \right]
$$

$$
= \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} R\left(\pi_{i,j} \,\|\, \mathcal{P}(a_1(T-t)) \times \mathcal{P}(a_2(T-t))\right) + (T-t)\left[c(q) - R(q\,\|\,a)\right].
$$

The mapping $q \to [c(q) - R(q\,\|\,a)]$ is convex if and only if $c(q) + h(q)$ is convex, and so convexity of $c(q) + h(q)$ is sufficient for the minimization problem to be convex in $(\pi_{i,j}, q_m)$. Note that in the deterministic case this condition holds with strict convexity, and that in the iid case $c(q) + h(q)$ is convex but never strictly convex (it is in fact always linear in q). Hence this is in a certain sense a borderline case, and one for which there may be nonuniqueness of minimizers. In the case of Markov coloring the condition may or may not hold–see [3] for further details.

This alternative rewriting of the objective function also has a practical benefit, in that the quantity to be minimized is now the sum of a convex function of π and a convex function of q, with no "cross terms." As a consequence, the formulas for $\pi_{i,j}$ and q_m also separate, and hence can be solved for explicitly in terms of the multipliers.

Having restricted already to the case where $c(q) + h(q)$ is convex, we now make a final restriction. To parallel the very explicit computations of the single color model, we need a specific form for c, and in particular a form that allows us to solve for the minimizers in terms of multipliers. This can be done when the rate function for the coloration has a representation in terms of relative entropy, which is the case for all the models introduced previously. The particular form we choose is $c(q) = bR(q\,\|\,a)$, where $b \in (1, \infty)$. The limit $b \uparrow \infty$ gives the deterministic coloration with parameters a_1 and a_2, and the limit $b \downarrow 1$ gives the iid coloration with parameters a_1 and a_2.

Define

$$
\mathcal{J}(x, t; y) \doteq \inf_{\substack{\varphi \in C\left([t,T], \mathcal{S}_{I,J}\right) \\ \varphi(t)=x, \varphi(T)=y}} \mathcal{I}(x, t; \varphi).
$$

Theorem 3.1. *Consider the allocation problem with either the deterministic or iid coloration process with parameters $a_1 > 0$ and $a_2 > 0$, an initial condition $(x, t) \in \mathcal{D}$, and a feasible terminal condition y. Then the quantity $\mathcal{J}(x, t; y)$ has the representation*

$$
\min_{(\pi, q) \in \mathcal{F}(x, t; y, T)} \left\{ \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} R\left(\pi_{i,j} \,\|\, \mathcal{P}(a_1(T-t)) \times \mathcal{P}(a_2(T-t))\right) + (T-t)(b-1)R(q\,\|\,a) \right\}.
$$

Proof. We will prove the representation for $b \in (1, \infty)$. Taking limits and using monotonicity in b will then establish the corresponding result for $b = 1$ and $b = \infty$. The same line of argument as in the single color case is followed. Hence we consider linear terminal conditions $F(y) = \langle y, \ell \rangle$ with $\ell = \ell_{i,j}, i = 0, \ldots, I+1, j = 0, \ldots, J+1$, and define

$$
\begin{aligned}
U(x, t) \quad = \quad \inf \Bigg\{ & \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} R\left(\pi_{i,j} \left\| \mathcal{P}(a_1(T-t)) \times \mathcal{P}(a_2(T-t))\right)\right) \\
& + (T-t)(b-1)R(q \,\|\, a) + \langle \ell, x \times \pi \rangle \Bigg\}.
\end{aligned}
$$

The infimum is over $\mathcal{F}(x, t, T)$, which is defined to be the set of all collections $(\pi_{i,j}, q_m)$ such that

$$
\sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} r_m \pi_{i,j}(r_1, r_2) = q_m(T-t), m = 1, 2
$$

and

$$
q_m \geq 0, m = 1, 2, q_1 + q_2 = 1.
$$

To study this problem define

$$
\begin{aligned}
&f(x, t; \pi, q) \\
&\doteq \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} R\left(\pi_{i,j} \left\| \mathcal{P}(a_1(T-t)) \times \mathcal{P}(a_2(T-t))\right)\right) + (T-t)(b-1)R(q \,\|\, a) + \langle \ell, x \times \pi \rangle,
\end{aligned}
$$

introduce Lagrange multipliers $\Lambda = (\lambda_{i,j}, i = 0, \ldots, I+1, j = 0, \ldots, J+1; \mu_m, m = 1, 2; \theta)$, and define

$$
\begin{aligned}
L(x, t; \Lambda, \pi, q) \quad \doteq \quad & f(x, t; \pi, q) + \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} \lambda_{i,j} x_{i,j} \left(1 - \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \pi_{i,j}(k, l)\right) \\
& + \sum_{m=1,2} \mu_m \left(q_m(T-t) - \sum_{i=0}^{I+1} \sum_{j=0}^{J+1} x_{i,j} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} r_m \pi_{i,j}(r_1, r_2)\right) \\
& + \theta(1 - q_1 - q_2).
\end{aligned}
$$

Analogously to the single color case,

$$
\pi_{i,j}(k, l; x, t; \Lambda) = \mathcal{P}_k(a_1(T-t))\mathcal{P}_l(a_2(T-t))e^{\lambda_{i,j}-1+k\mu_1+l\mu_2-\ell_{i+k,j+l}}.
$$

The equation for q is

$$
(T-t)(b-1)\left[\log\left(\frac{q_m}{a_m}\right) + 1\right] + \mu_m(T-t) - \theta = 0,
$$

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so that
$$q_m(x,t;\Lambda) = a_m e^{-\frac{\mu_m}{b-1}} e^{\frac{\theta}{(T-t)(b-1)}} e^{-1}.$$

For $i = 0, \ldots, I, I+1, j = 0, \ldots, J, J+1$ and $m = 1, 2$ let

$$
\begin{aligned}
G_{i,j}(x,t;\Lambda) &= \left(1 - \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \pi_{i,j}(k,l;x,t;\Lambda)\right), \\
G_m(x,t;\Lambda) &= \left(q_m(x,t;\Lambda)(T-t) - \sum_{i=0}^{I+1}\sum_{j=0}^{J+1} x_{i,j} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} r_m \pi_{i,j}(k,l;x,t;\Lambda)\right), \\
G_3(x,t;\Lambda) &= (1 - q_1(x,t;\Lambda) - q_2(x,t;\Lambda)).
\end{aligned}
$$

When discussing uniqueness of the multipliers we must work with a matrix indexed by the subscripts of these functions, and the particular ordering of the i, j as subscripts is unimportant.

We next present three lemmas that are analogues of ones proved in the case of a single color. Since the proofs of the first two are also direct analogues they are omitted.

Lemma 3.2 (General properties). *For any $(x,t) \in \mathcal{D}$, $\mathcal{F}(x,t;T)$ is nonempty, minimizing measures π^* exist, and if $x_{i,j} > 0$ then $\pi^*_{i,j}(k,l) > 0$ for all k and l.*

Lemma 3.3 (Characterization of the minimizer). *For any $(x,t) \in \mathcal{D}$ there exists $\Lambda \in \mathbb{R}^{(I+2)\times(J+2)+3}$ so that $G(x,t;\Lambda) = 0$, and $\pi_{i,j}(k,l;x,t;\Lambda)$, $q_m(x,t;\Lambda)$ is a minimizer in the definition of $U(x,t)$.*

Lemma 3.4 (Uniqueness of characterization). *For $(x,t) \in \mathcal{D}$, there is only one $\Lambda \in \mathbb{R}^{(I+2)\times(J+2)+3}$ such that $G(x,t,\Lambda) = 0$.*

Proof. We have

$$
\begin{cases}
\frac{\partial \pi_{i,j}(k,l;x,t;\Lambda)}{\partial \lambda_{i,j}} &= \pi_{i,j}(k,l;x,t;\Lambda) \\
\frac{\partial \pi_{i,j}(k,l;x,t;\Lambda)}{\partial \mu_1} &= k\pi_{i,j}(k,l;x,t;\Lambda) \\
\frac{\partial \pi_{i,j}(k,l;x,t;\Lambda)}{\partial \mu_2} &= l\pi_{i,j}(k,l;x,t;\Lambda) \\
\frac{\partial q_m(x,t;\Lambda)}{\partial \mu_m} &= -\frac{1}{b-1}q_m(x,t;\Lambda) \\
\frac{\partial q_m(x,t;\Lambda)}{\partial \theta} &= \frac{1}{(T-t)(b-1)}q_m(x,t;\Lambda),
\end{cases}
$$

and all other partial derivatives are zero. As in the single color case it is enough to show the negative definiteness of $D_\Lambda G$. Using a suitable dominating function to

justify the interchange of differentiation and summation and the definitions

$$\alpha_{i,j} = \sum_{k,l=0}^{\infty} \pi_{i,j}(k,l;x,t;\Lambda)$$

$$T_m^{i,j} = \sum_{r_1,r_2=0}^{\infty} r_m \pi_{i,j}(r_1,r_2;x,t;\Lambda)$$

$$C_{1,1} = \sum_{i=0}^{I+1}\sum_{j=0}^{J+1} x_{i,j} \sum_{r_1,r_2=0}^{\infty} r_1^2 \pi_{i,j}(r_1,r_2;x,t;\Lambda)$$

$$C_{2,2} = \sum_{i=0}^{I+1}\sum_{j=0}^{J+1} x_{i,j} \sum_{r_1,r_2=0}^{\infty} r_2^2 \pi_{i,j}(r_1,r_2;x,t;\Lambda)$$

$$C_{1,2} = \sum_{i=0}^{I+1}\sum_{j=0}^{J+1} x_{i,j} \sum_{r_1,r_2=0}^{\infty} r_1 r_2 \pi_{i,j}(r_1,r_2;x,t;\Lambda),$$

$D_\Lambda G$ equals [with $q_m = q_m(x,t;\Lambda)$]

$$\begin{pmatrix} -\alpha_{0,0} & \cdots & 0 & -T_1^{0,0} & -T_2^{0,0} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -\alpha_{I+1,J+1} & -T_1^{I+1,j+1} & -T_2^{I+1,j+1} & 0 \\ -x_{0,0}T_1^{0,0} & \cdots & -x_{I+1,J+1}T_1^{I+1,J+1} & -\frac{T-t}{b-1}q_1 - C_{1,1} & -C_{1,2} & -\frac{1}{b-1}q_1 \\ -x_{0,0}T_2^{0,0} & \cdots & -x_{I+1,J+1}T_2^{I+1,J+1} & -C_{1,2} & -\frac{T-t}{b-1}q_2 - C_{2,2} & -\frac{1}{b-1}q_2 \\ 0 & \cdots & 0 & \frac{1}{b-1}q_1 & \frac{1}{b-1}q_2 & -\frac{q_1+q_2}{(T-t)(b-1)} \end{pmatrix}.$$

Since diagonalizing all save the lower right 3×3-submatrix produces strictly negative values on the diagonal, we need only check the negative definiteness of

$$\begin{pmatrix} \sum_{i,j}\frac{x_{i,j}}{\alpha_{i,j}}(T_1^{i,j})^2 - \frac{T-t}{b-1}q_1 - C_{1,1} & \sum_{i,j}\frac{x_{i,j}}{\alpha_{i,j}}T_1^{i,j}T_2^{i,j} - C_{1,2} & -\frac{1}{b-1}q_1 \\ \sum_{i,j}\frac{x_{i,j}}{\alpha_{i,j}}T_1^{i,j}T_2^{i,j} - C_{1,2} & \sum_{i,j}\frac{x_{i,j}}{\alpha_{i,j}}(T_2^{i,j})^2 - \frac{T-t}{b-1}q_2 - C_{2,2} & -\frac{1}{b-1}q_2 \\ \frac{1}{b-1}q_1 & \frac{1}{b-1}q_2 & -\frac{q_1+q_2}{(T-t)(b-1)} \end{pmatrix}$$

Since

$$\begin{pmatrix} -\frac{T-t}{b-1}q_1 & 0 & -\frac{1}{b-1}q_1 \\ 0 & -\frac{T-t}{b-1}q_2 & -\frac{1}{b-1}q_2 \\ \frac{1}{b-1}q_1 & \frac{1}{b-1}q_2 & -\frac{q_1+q_2}{(T-t)(b-1)} \end{pmatrix}$$

is obviously negative definite, we need only check the 2×2 matrix

$$\begin{pmatrix} \sum_{i,j}\frac{x_{i,j}}{\alpha_{i,j}}(T_1^{i,j})^2 - C_{1,1} & \sum_{i,j}\frac{x_{i,j}}{\alpha_{i,j}}T_1^{i,j}T_2^{i,j} - C_{1,2} \\ \sum_{i,j}\frac{x_{i,j}}{\alpha_{i,j}}T_1^{i,j}T_2^{i,j} - C_{1,2} & \sum_{i,j}\frac{x_{i,j}}{\alpha_{i,j}}(T_2^{i,j})^2 - C_{2,2} \end{pmatrix}.$$

However, letting $\pi_{i,j}(r_1,r_2) = \pi_{i,j}(r_1,r_2;x,t;\Lambda)$ and pre- and post-multiplying by the nonzero vector (z_1,z_2) produces

$$\sum_{i,j}\frac{x_{i,j}}{\alpha_{i,j}}\left(\sum_{r_1,r_2}(z_1 r_1 + z_2 r_2)\pi_{i,j}(r_1,r_2) - \sum_{r_1,r_2}(z_1 r_1 + z_2 r_2)^2 \pi_{i,j}(r_1,r_2) \cdot \sum_{r_1,r_2}\pi_{i,j}(r_1,r_2)\right) \le 0.$$

Thus the entire matrix is negative definite. $\qquad\square$

Since we have restricted to the case of MB there is no analogue of the non-smooth function $\tau(x,t)$, and hence the existence of a smooth extension of U to a neighborhood of $\mathcal{D}$ follows directly from the implicit function theorem. The next result expresses the derivatives in terms of the multipliers.

Theorem 3.5. *Fix $(x,t) \in D$, and let Λ^* be the associated Lagrange multiplier. Then*

$$D_{x_{i,j}} U(x,t) = (\lambda_{i,j}^* - 1)$$

and

$$D_t U(x,t) = (b-1) - \frac{\theta^*}{(T-t)}.$$

Proof. With

$$H(x,t;\Lambda) \doteq L(x,t;\Lambda,\pi_{i,j}(\cdot,x,t;\Lambda), q_m(x,t;\Lambda))$$

we can write

$$U(x,t) = H(x,t;\Lambda(x,t)),$$

where $\Lambda(x,t)$ is the unique solution to the constraint equations. Then

$$D_{x_{i,j}} U(x,t) = D_{x_{i,j}} H(x,t;\Lambda^*) + D_\Lambda H(x,t;\Lambda^*) D_{x_{i,j}} \Lambda(x,t).$$

As in the single color case $D_\Lambda H(x,t;\Lambda^*) = 0$, and so $D_{x_{i,j}} U(x,t) = D_{x_{i,j}} H(x,t;\Lambda^*)$, and an analogous argument also gives $D_t U(x,t) = D_t H(x,t;\Lambda^*)$. Note that both $\pi_{i,j}(k,l;x,t;\Lambda)$ and $q_m(x,t;\Lambda)$ are actually independent of x, and hence can be written $\pi_{i,j}(k,l;t;\Lambda)$ and $q_m(t;\Lambda)$. Keeping in mind the t dependence but temporarily suppressing both t and Λ^* in the notation, we will use

$$(T-t)(b-1)R(q\,\|\,a) = -(T-t)\left[q_1\mu_1^* + q_2\mu_2^*\right] + \left[\theta^* - (T-t)(b-1)\right](q_1+q_2).$$

The quantity to be differentiated is thus

$$\sum_{i=0}^{I+1}\sum_{j=0}^{J+1} x_{i,j} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \left[\lambda_{i,j}^* - 1 + k\mu_1^* + l\mu_2^*\right]\pi_{i,j}(k,l) - (T-t)\left[q_1\mu_1^* + q_2\mu_2^*\right]$$

$$+[\theta^* - (T-t)(b-1)](q_1+q_2)$$

$$+\sum_{i=0}^{I+1}\sum_{j=0}^{J+1} \lambda_{i,j}^* x_{i,j}\left(1 - \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\pi_{i,j}(k,l)\right)$$

$$+\sum_{m=1,2}\mu_m^*\left(q_m(T-t) - \sum_{i=0}^{I+1}\sum_{j=0}^{J+1}x_{i,j}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}r_m\pi_{i,j}(r_1,r_2)\right)$$

$$+\theta^*(1 - q_1 - q_2)$$

$$= -\sum_{i=0}^{I+1}\sum_{j=0}^{J+1}x_{i,j}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\pi_{i,j}(k,l) - (T-t)(b-1)(q_1+q_2) + \sum_{i=0}^{I+1}\sum_{j=0}^{J+1}\lambda_{i,j}^* x_{i,j} + \theta^*.$$

Since

$$
\begin{aligned}
\frac{\partial \pi_{i,j}(k,l;t;\Lambda)}{\partial t} &= e^{\lambda_{i,j}-1+k\mu_1+l\mu_2-\ell_{i+k,j+l}}\frac{\partial \mathcal{P}_k(a_1(T-t))\mathcal{P}_l(a_2(T-t))}{\partial t} \\
&= \frac{e^{\lambda_{i,j}-1+k\mu_1+l\mu_2-\ell_{i+k,j+l}}}{k!l!}\frac{\partial e^{-(T-t)}a_1^k(T-t)^k a_2^l(T-t)^l}{\partial t} \\
&= \left[1-\frac{k}{T-t}-\frac{l}{T-t}\right]\pi_{i,j}(k,l;t;\Lambda)
\end{aligned}
$$

and

$$
\frac{\partial q_m(t;\Lambda)}{\partial t} = \left[\frac{\theta}{(T-t)^2(b-1)}\right]q_m(t;\Lambda),
$$

it follows that

$$
D_{x_{i,j}}U(x,t) = (\lambda_{i,j}^* - 1)
$$

and

$$
\begin{aligned}
D_t U(x,t) &= -\sum_{i=0}^{I+1}\sum_{j=0}^{J+1}x_{i,j}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[1-\frac{k}{T-t}-\frac{l}{T-t}\right]\pi_{i,j}(k,l) \\
&\quad +(b-1)\,(q_1+q_2)-\frac{\theta^*}{(T-t)}\,(q_1+q_2) \\
&= -1+q_1+q_2+(b-1)-\frac{\theta^*}{(T-t)} \\
&= (b-1)-\frac{\theta^*}{(T-t)}.
\end{aligned}
$$

$\square$

For the particular rate function of interest here,

$$
\begin{aligned}
b(\gamma) &= \sup\left[\langle\gamma,q\rangle - bR(q\,\|\,a)\right] \\
&= -b\inf\left[-\frac{1}{b}\langle\gamma,q\rangle + R(q\,\|\,a)\right] \\
&= b\log\left(e^{\frac{\gamma_1}{b}}a_1 + e^{\frac{\gamma_2}{b}}a_2\right).
\end{aligned}
$$

Hence by (3.1), the PDE to be satisfied by U takes the form

$$
\begin{aligned}
W_t - b\log\Bigg(& \left(\sum_{i,j,i\leq I}e^{-(W_{x_{i+1,j}}-W_{x_{i,j}})}x_{i,j} + \sum_{i,j,i=I+1}x_{i,j}\right)^{\frac{1}{b}}a_1 \\
& +\left(\sum_{i,j,j\leq J}e^{-(W_{x_{i,j+1}}-W_{x_{i,j}})}x_{i,j} + \sum_{i,j,j=J+1}x_{i,j}\right)^{\frac{1}{b}}a_2\Bigg) = 0. \qquad (3.2)
\end{aligned}
$$

Theorem 3.6. U *satisfies (3.2) on* $\mathcal{D}$.

Proof. By Theorem 3.5, this will be true if

$$(b-1) - \frac{\theta^*}{(T-t)} - b \log\left(\left(\sum_{i,j,i\leq I} e^{-(\lambda^*_{i+1,j}-\lambda^*_{i,j})} x_{i,j} + \sum_{i,j,i=I+1} x_{i,j}\right)^{\frac{1}{b}} a_1\right.$$

$$\left. + \left(\sum_{i,j,j\leq J} e^{-(\lambda^*_{i,j+1}-\lambda^*_{i,j})} x_{i,j} + \sum_{i,j,j=J+1} x_{i,j}\right)^{\frac{1}{b}} a_2\right) = 0. \qquad (3.3)$$

For $i \leq I$

$$\frac{\pi_{i+1,j}(k,l)}{\pi_{i,j}(k+1,l)} = \frac{e^{\lambda_{i+1,j}-\lambda_{i,j}} e^{-\mu_1}(k+1)}{a_1(T-t)}.$$

Summing on k and l gives

$$e^{\lambda_{i,j}-\lambda_{i+1,j}} = \frac{e^{-\mu_1}}{a_1(T-t)} \sum_{r_1,r_2} r_1 \pi_{i,j}(r_1,r_2),$$

and the analogous formula

$$e^{\lambda_{i,j}-\lambda_{i,j+1}} = \frac{e^{-\mu_2}}{a_2(T-t)} \sum_{r_1,r_2} r_2 \pi_{i,j}(r_1,r_2)$$

applies for $j \leq J$. We also have

$$1 = \frac{e^{-\mu_2}}{a_2(T-t)} \sum_{r_1,r_2} r_2 \pi_{I+1,j}(r_1,r_2) = \frac{e^{-\mu_2}}{a_2(T-t)} \sum_{r_1,r_2} r_2 \pi_{i,J+1}(r_1,r_2)$$

if $j \leq J+1$ or $i \leq I+1$. Hence

$$\sum_{i,j,i\leq I} e^{-(\lambda^*_{i+1,j}-\lambda^*_{i,j})} x_{i,j} + \sum_{i,j,i=I+1} x_{i,j} = e^{-\mu^*_1}\frac{q_1}{a_1},$$

$$\sum_{i,j,j\leq J} e^{-(\lambda^*_{i,j+1}-\lambda^*_{i,j})} x_{i,j} + \sum_{i,j,j=J+1} x_{i,j} = e^{-\mu^*_2}\frac{q_2}{a_2}.$$

Now $q_1 + q_2 = 1$ implies

$$\begin{aligned}
0 &= \log\left(q_1(t;\Lambda) + q_2(t;\Lambda)\right) \\
&= \log\left(a_1 e^{-\frac{\mu_1}{b-1}} e^{\frac{\theta}{(T-t)(b-1)}-1} + a_2 e^{-\frac{\mu_2}{b-1}} e^{\frac{\theta}{(T-t)(b-1)}-1}\right) \\
&= \log\left(a_1 e^{-\frac{\mu_1}{b-1}} + a_2 e^{-\frac{\mu_2}{b-1}}\right) + \frac{\theta}{(T-t)(b-1)} - 1.
\end{aligned}$$

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The left hand side of (3.3) becomes

$$(b-1) - \frac{\theta^*}{(T-t)} - b\log\left(\left(e^{-\mu_1^* }q_1/a_1\right)^{\frac{1}{b}}a_1 + \left(e^{-\mu_2^*}q_2/a_2\right)^{\frac{1}{b}}a_2\right)$$

$$= (b-1) - \frac{\theta^*}{(T-t)} - b\log\left(\left(e^{-\mu_1^*}e^{-\frac{\mu_1^*}{b-1}}e^{\frac{\theta^*}{(T-t)(b-1)}-1}\right)^{\frac{1}{b}}a_1 + \left(e^{-\mu_2^*}e^{-\frac{\mu_2^*}{b-1}}e^{\frac{\theta^*}{(T-t)(b-1)}-1}\right)^{\frac{1}{b}}a_2\right)$$

$$= -(b-1) + \frac{\theta^*}{(T-t)} - b\left[\log\left(\left(e^{-\frac{b\mu_1^*}{b-1}}\right)^{\frac{1}{b}}a_1 + \left(e^{-\frac{b\mu_2^*}{b-1}}\right)^{\frac{1}{b}}a_2\right) + \frac{\theta^*}{b(T-t)(b-1)} - \frac{1}{b}\right]$$

$$= -b + \frac{b\theta^*}{(T-t)(b-1)} - \frac{b\theta^*}{(T-t)(b-1)} + b$$

$$= 0,$$

and the theorem is proved. $\qquad\square$

3.4 Minimizing Trajectories

We end this section by stating without proof the form of the minimizing trajectories. As in the case of a single color we consider only the empty initial condition. In contrast with that case, here the minimizing q must be determined first via Lagrange multipliers. Once q is given, we define

$$\varphi_{0,0}(t_1, t_2)$$
$$\doteq C\mathcal{P}_0(\rho q_1 t_1)\mathcal{P}_0(\rho q_2 t_2) + \sum_{k=0}^{I}\sum_{l=0}^{J}(y_{k,l} - C\mathcal{P}_k(\rho q_1 T)\mathcal{P}_l(\rho q_2 T))\left(1 - \frac{t_1}{T}\right)^k\left(1 - \frac{t_2}{T}\right)^l$$

and

$$\varphi_{i,j}(t_1, t_2) \doteq \frac{(-t_1)^i}{i!}\frac{(-t_2)^j}{j!}\varphi_{0,0}^{(i,j)}(t_1, t_2).$$

In terms of these functions we set

$$\varphi_{i,j}(t) \doteq \varphi_{i,j}(t, t).$$

and for $i \le I$ and $j \le J$

$$\varphi_{I+,j}(t) = \sum_{i=I+1}^{\infty}\varphi_{i,j}(t),\ \varphi_{i,J+}(t) = \sum_{j=J+1}^{\infty}\varphi_{i,j}(t),\ \varphi_{I+J+}(t) = \sum_{i=I+1}^{\infty}\sum_{j=J+1}^{\infty}\varphi_{i,j}(t).$$

With q determined via Lagrange multipliers, the parameters $\rho > 0$ and $C \ge 0$ are chosen so that

$$\frac{\rho T - \sum_{i=0}^{I}\sum_{j=0}^{J}(i+j)\mathcal{P}_i(\rho q_1 T)\mathcal{P}_j(\rho q_2 T)}{1 - \sum_{i=0}^{I}\sum_{j=0}^{J}\mathcal{P}_i(\rho q_1 T)\mathcal{P}_j(\rho q_2 T)} = \frac{T - \sum_{i=0}^{I}\sum_{j=0}^{J}(i+j)y_{i,j}}{1 - \sum_{i=0}^{I}\sum_{j=0}^{J}y_{i,j}}$$

and

$$C \doteq \frac{1 - \sum_{i=0}^{I}\sum_{j=0}^{J}y_{i,j}}{1 - \sum_{i=0}^{I}\sum_{j=0}^{J}\mathcal{P}_i(\rho q_1 T)\mathcal{P}_j(\rho q_2 T)} = \frac{T - \sum_{i=0}^{I}\sum_{j=0}^{J}(i+j)y_{i,j}}{\rho T - \sum_{i=0}^{I}\sum_{j=0}^{J}(i+j)\mathcal{P}_i(\rho q_1 T)\mathcal{P}_j(\rho q_2 T)}.$$

The proof that these trajectories achieve the minimal cost parallels that of the single color case, with an appropriate modification of the notion of completely monotone that is suitable for functions of two independent variables.

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